Advanced Vibration
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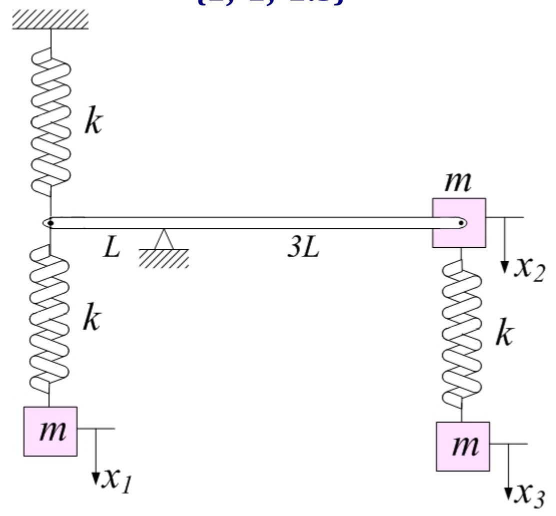
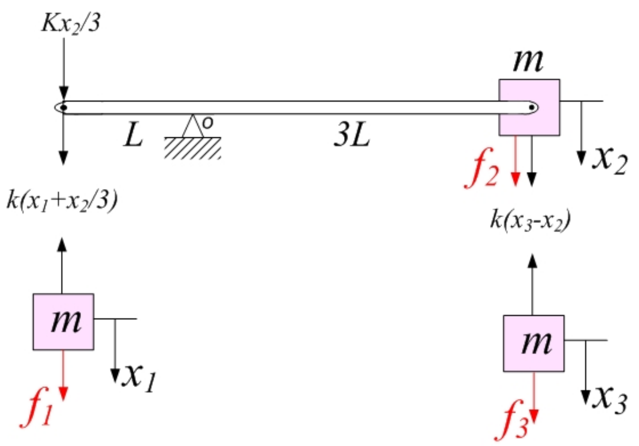
Homework10

1. **Consider the system shown below included 3 masses and a rod with negligible mass. Find:**
    - **Stiffness coefficients with stiffness method**
    - **Fundamental natural frequency by using Rayleigh's method and the mode shape below:**

$\{1,-1,-1.5\}^T$





Stiffness coefficients:

$$f_i = k_{ij} x_j$$

Static Equilibrium Equations:

$$\sum F_{x1} = 0 \Rightarrow f_1 = k(x_1 + \frac{x_2}{3})$$

$$\sum M_0 = 0 \Rightarrow f_2 = k(\frac{x_1}{3} + \frac{11}{9} x_2 - x_3)$$

$$\sum F_{x3} = 0 \Rightarrow f3 = k(x_3 - x_2)$$

- ***Case1: x_1=1 , x_2=x_3=0***

$$f_1 = k = k_{11}$$
$$f_2 = \frac{k}{3} = k_{21}$$
$$f_3 = 0 = k_{31}$$

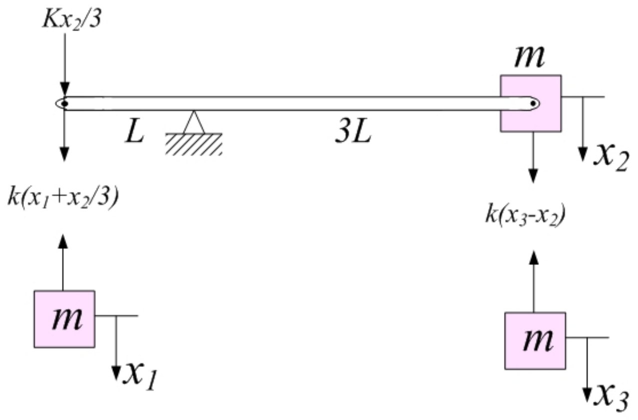
- ***Case1: x_2=1 , x_1=x_3=0***

$$f_1 = \frac{k}{3} = k_{12}$$
$$f_2 = \frac{11k}{9} = k_{22}$$
$$f_3 = -k = k_{32}$$

- ***Case1: x_3=1 , x_1=x_2=0***

$$f_1 = 0 = k_{13}$$
$$f_2 = -k = k_{23}$$
$$f_3 = k = k_{33}$$

$$\underline{\underline{K}} = k\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{11}{9} & -1 \\ 0 & -1 & 1 \end{bmatrix}$$



Equations of motion:

$$\sum F_{x1} = m\ddot{x}_1 \Rightarrow m\ddot{x}_1 + k(\frac{x_2}{3} + x_1) = 0$$

$$\sum M_0 = m\ddot{x}_2 \times 3L \Rightarrow k(x_3 - x_2) \times 3L - k(\frac{2x_2}{3} + x_1) \times L \Rightarrow m\ddot{x}_2 + \frac{11}{9}k\,x_2 + \frac{k}{3}x_1 - k\,x_3 = 0$$

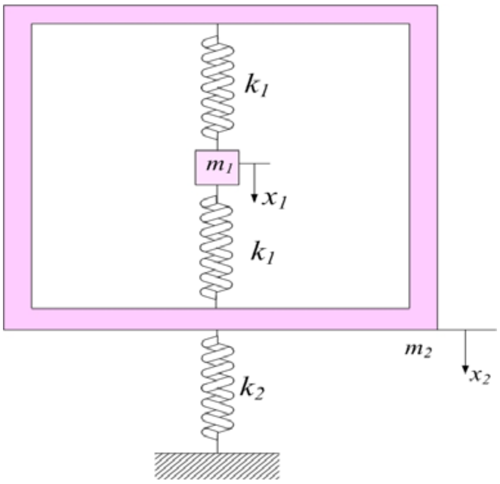
$$\sum F_{x3} = m\ddot{x}_3 \Rightarrow m\ddot{x}_3 + k(x_3 - x_2) = 0$$

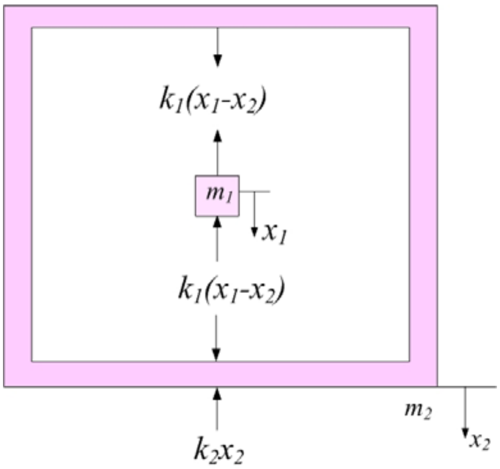
$$\Rightarrow \underline{\underline{M}} = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \omega^2 = \frac{\underline{X}^T \underline{\underline{K}}\,\underline{X}}{\underline{X}^T \underline{\underline{M}}\,\underline{X}} \quad X^T = \{1, -1, -1.5\}^T \Rightarrow \omega = 0.435365\sqrt{\frac{k}{m}}$$

2. The system shown below is a packaging model of fragile bodies. If m_1=m_2=m and k_1=k_2=k ,determine :

- ***Differential equations of motion***
- ***Fundamental natural frequency by using Rayleigh's method and the mode shape below:***

$$\{1,1.5\}^T$$





F.B.D

Differential Equations of motion:

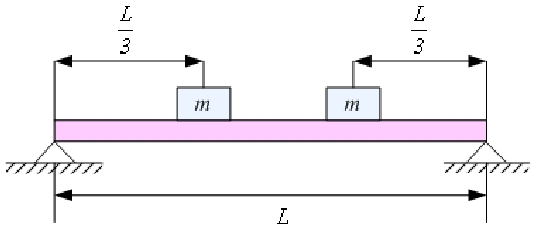
$$\sum F_{x1} = m_1\ddot{x}_1 \Rightarrow m_1\ddot{x}_1 + 2k(x_1 - x_2) = 0$$

$$\sum F_{x2} = m_2\ddot{x}_2 \Rightarrow m_2\ddot{x}_2 + 3kx_2 - 2kx_1 = 0$$

$$k_1 = k_2 = k\ ,\ m_1 = m_2 = m \Rightarrow \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\underline{\ddot{x}} + \begin{bmatrix} 2k & -2k \\ -2k & 3k \end{bmatrix}\underline{x} = 0$$

$$\underline{\underline{M}} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \underline{\underline{K}} = \begin{bmatrix} 2k & -2k \\ -2k & 3k \end{bmatrix} \Rightarrow \omega^2 = \frac{\underline{X}^T \underline{\underline{K}}\,\underline{X}}{\underline{X}^T \underline{\underline{M}}\,\underline{X}} \quad X^T = \{1,1.5\}^T \Rightarrow \omega = 0.91987\sqrt{\frac{k}{m}}$$

3. Consider a beam shown below pinned at two ends and consist of two attached mass. Determine the first and second natural frequencies of the beam by using natural frequencies and mode shapes of the beam pinned at two end .The mass of the beam is M.



$$a_1=\frac{L}{3},a_2=\frac{2L}{3}\quad,\quad \ddot q_i+\omega_i^2q_i=\frac{1}{M_i}\left[-m\phi_i(a_1)\sum_j\ddot q_j\phi_j(a_1)-m\phi_i(a_2)\sum_j\ddot q_j\phi_j(a_2)\right]\Rightarrow$$

$$q_i=\bar q_ie^{i\omega t}$$

$$\Rightarrow \ddot q_1+\omega_1^2q_1=\frac{1}{M_1}\left[-m\phi_1(a_1)[\ddot q_1\phi_1(a_1)+\ddot q_2\phi_2(a_1)]-m\phi_1(a_2)[\ddot q_1\phi_1(a_2)+\ddot q_2\phi_2(a_2)]\right]\Rightarrow$$

$$\bar q_1=\frac{m\omega^2}{M_1(\omega_1^2-\omega^2)}\left[\phi_1(a_1)[\bar q_1\phi_1(a_1)+\bar q_2\phi_2(a_1)]+\phi_1(a_2)[\bar q_1\phi_1(a_2)+\bar q_2\phi_2(a_2)]\right]$$

$$\ddot q_2+\omega_2^2q_2=\frac{1}{M_2}\left[-m\phi_2(a_1)[\ddot q_1\phi_1(a_1)+\ddot q_2\phi_2(a_1)]-m\phi_2(a_2)[\ddot q_1\phi_1(a_2)+\ddot q_2\phi_2(a_2)]\right]\Rightarrow$$

$$\bar q_2=\frac{m\omega^2}{M_2(\omega_2^2-\omega^2)}\left[\phi_2(a_1)[\bar q_1\phi_1(a_1)+\bar q_2\phi_2(a_1)]+\phi_2(a_2)[\bar q_1\phi_1(a_2)+\bar q_2\phi_2(a_2)]\right]$$

$$\phi_i(x)=\sqrt{2}\sin\frac{i\pi x}{L}\Rightarrow\phi_1(x)=\sqrt{2}\sin\frac{\pi x}{L},\phi_2(x)=\sqrt{2}\sin\frac{2\pi x}{L}\Rightarrow\begin{cases}\phi_1(a_1)=\phi_1(a_2)=\phi_2(a_1)=\sqrt{\frac{3}{2}}\\ \phi_2(a_2)=-\sqrt{\frac{3}{2}}\end{cases}$$

$$M_1=M_2=M\quad,\quad \omega_i=\left(\frac{i\pi}{L}\right)^2\sqrt{\frac{EI}{\frac{M}{L}}}=\left(\frac{i\pi}{L}\right)^2\sqrt{\frac{EIL}{M}}\Rightarrow\begin{cases}\omega_1=\left(\frac{\pi}{L}\right)^2\sqrt{\frac{EIL}{M}}\\ \omega_2=4\left(\frac{\pi}{L}\right)^2\sqrt{\frac{EIL}{M}}\end{cases}$$

$$\begin{cases}\left(1-\frac{3m\omega^2}{M(\omega_1^2-\omega^2)}\right)\bar q_1+0\times\bar q_2=0\\ 0\times\bar q_1+\left(1-\frac{3m\omega^2}{M(\omega_2^2-\omega^2)}\right)\bar q_2=0\end{cases}\Rightarrow\begin{bmatrix}\left(1-\frac{3m\omega^2}{M(\omega_1^2-\omega^2)}\right) & 0\\ 0 & \left(1-\frac{3m\omega^2}{M(\omega_2^2-\omega^2)}\right)\end{bmatrix}\begin{Bmatrix}\bar q_1\\ \bar q_2\end{Bmatrix}=0\Rightarrow$$

$$\det\left(\begin{bmatrix}\left(1-\frac{3m\omega^2}{M(\omega_1^2-\omega^2)}\right) & 0\\ 0 & \left(1-\frac{3m\omega^2}{M(\omega_2^2-\omega^2)}\right)\end{bmatrix}\right)=0\Rightarrow\left(1-\frac{3m\omega^2}{M(\omega_1^2-\omega^2)}\right)\left(1-\frac{3m\omega^2}{M(\omega_2^2-\omega^2)}\right)=0$$

$$\Rightarrow\omega_1=\frac{\pi^2}{L^{1.5}}\sqrt{\frac{EI}{3m+M}},\omega_2=\frac{4\pi^2}{L^{1.5}}\sqrt{\frac{EI}{3m+M}}$$