Review

Characteristic Equation From State Space Equation

$$[sI-A]^{-1}=\phi(s)$$

Characteristic Equation: det(sI-A)=0

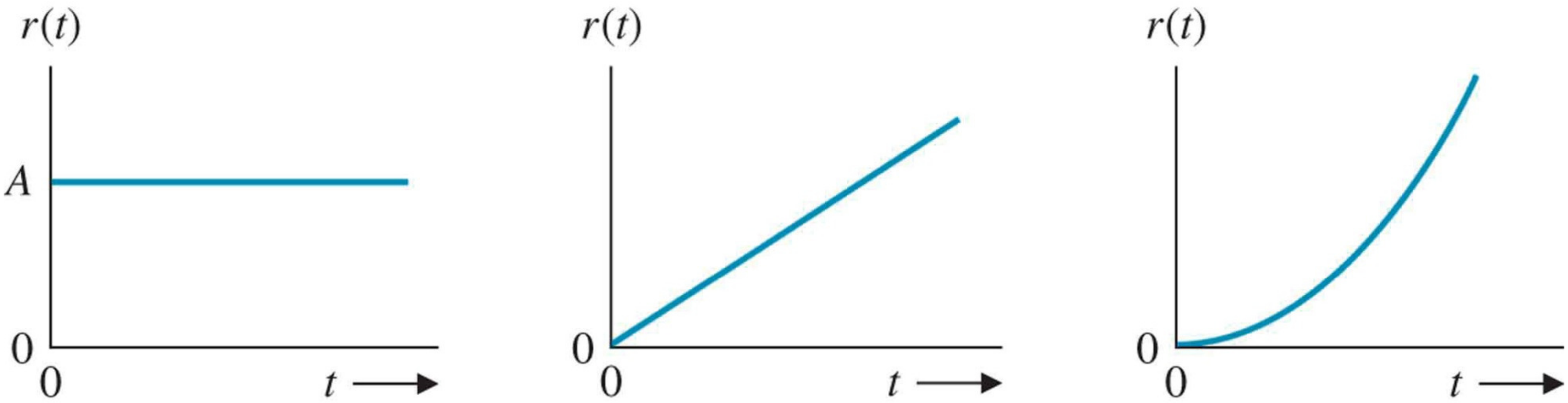
Chapter 4
Feedback Control System Characteristics

- **Transient and Steady-State Response Analysis**
- **Sensitivity to Model Uncertainties**
- **Steady-state Errors ($t \rightarrow \infty$): $Y(\infty)$, $E(\infty)$**
- **Disturbance Rejection**

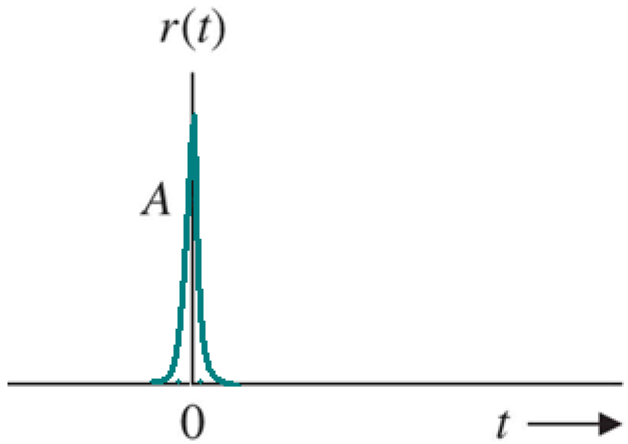
Typical Test Signals

- Step functions (*)
- Ramp functions (*)
- Impulse functions (*)
- Parabolic functions
- Sinusoidal functions (Later in frequency analysis)
- White noise

Test signals



(a) (b) (c)



Transient and Steady-State Response

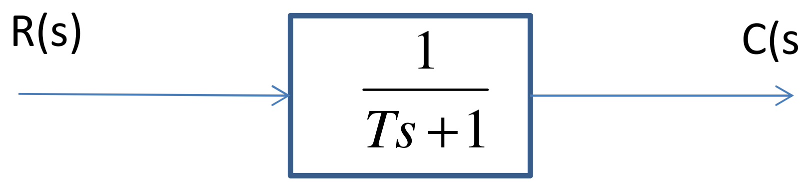
Time response has two parts:

1) Transient Response: System response from initial state to final state
2) Steady-state Response: System response when ***t*** approaches infinity.

$$c(t) = C_{tr}(t) + C_{ss}(t)$$

First Order Systems

$$c(t) = C_{tr}(t) + C_{ss}(t)$$



$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$

Unit step response of first order systems

$$C(s) = G(s)R(s)$$

$$C(s) = \frac{1}{Ts+1} L\{u(t)\}$$

$$= \frac{1}{Ts+1}\frac{1}{s} = \frac{A}{s} + \frac{B}{Ts+1}$$

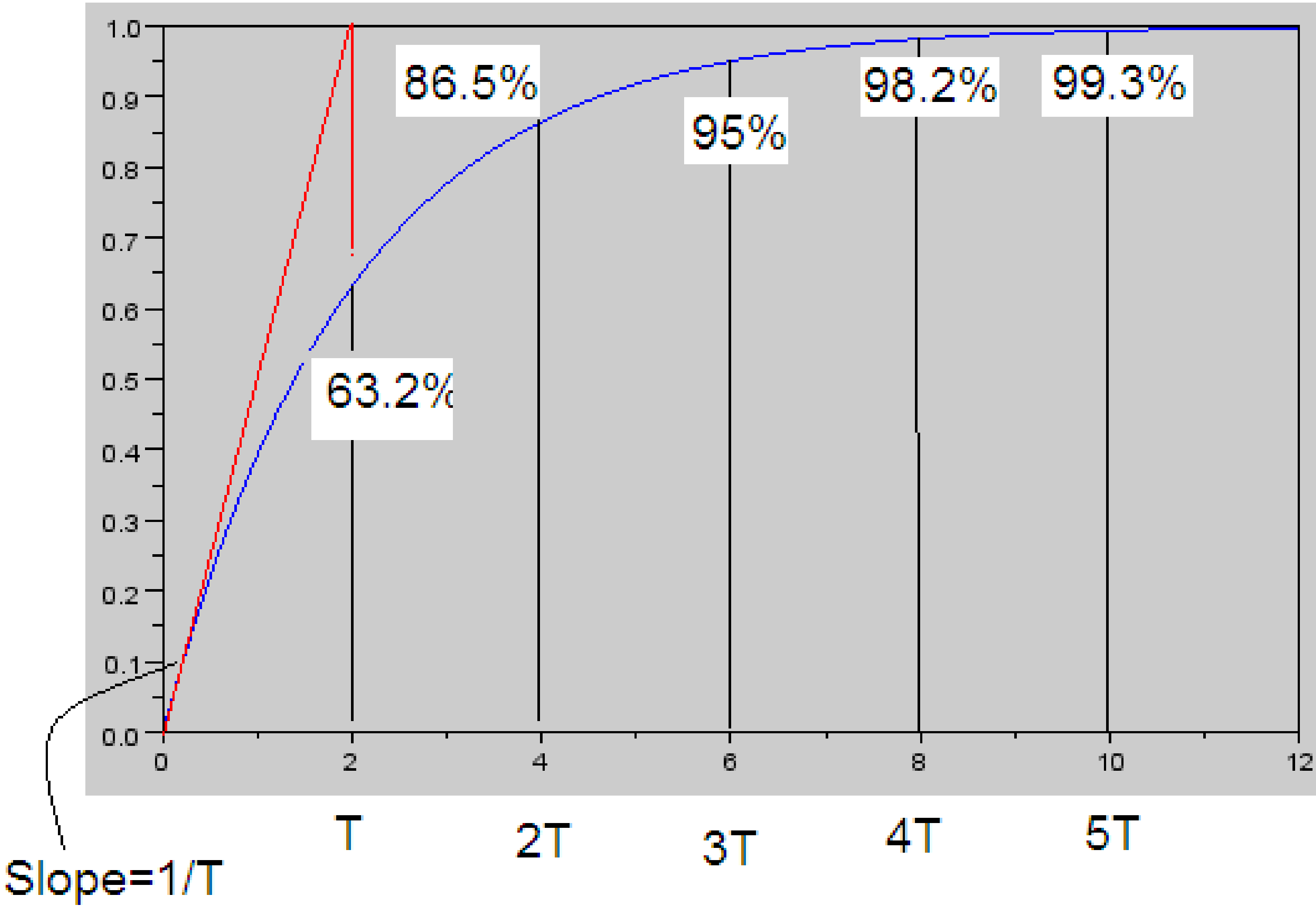
$$= \frac{1}{\mathrm{s}} - \frac{T}{Ts+1} = \frac{1}{\mathrm{s}} - \frac{1}{s+(1/T)}$$

$$c(t) = 1 - e^{-\frac{t}{T}} \quad \text{for} \quad t \geq 0 \qquad (1)$$

From Eq (1),

At $t = 0$, $c(t) = 0$

$At\ t \rightarrow \infty, \quad c(t) = 1$



$$c(t) = 1 - e^{-t/T}$$

$$Slope, \quad \left.\frac{dc(t)}{dt}\right|_{t=0} = 0 - \left.\frac{-1}{T} e^{-t/T}\right|_{t=0} = \frac{1}{T}$$

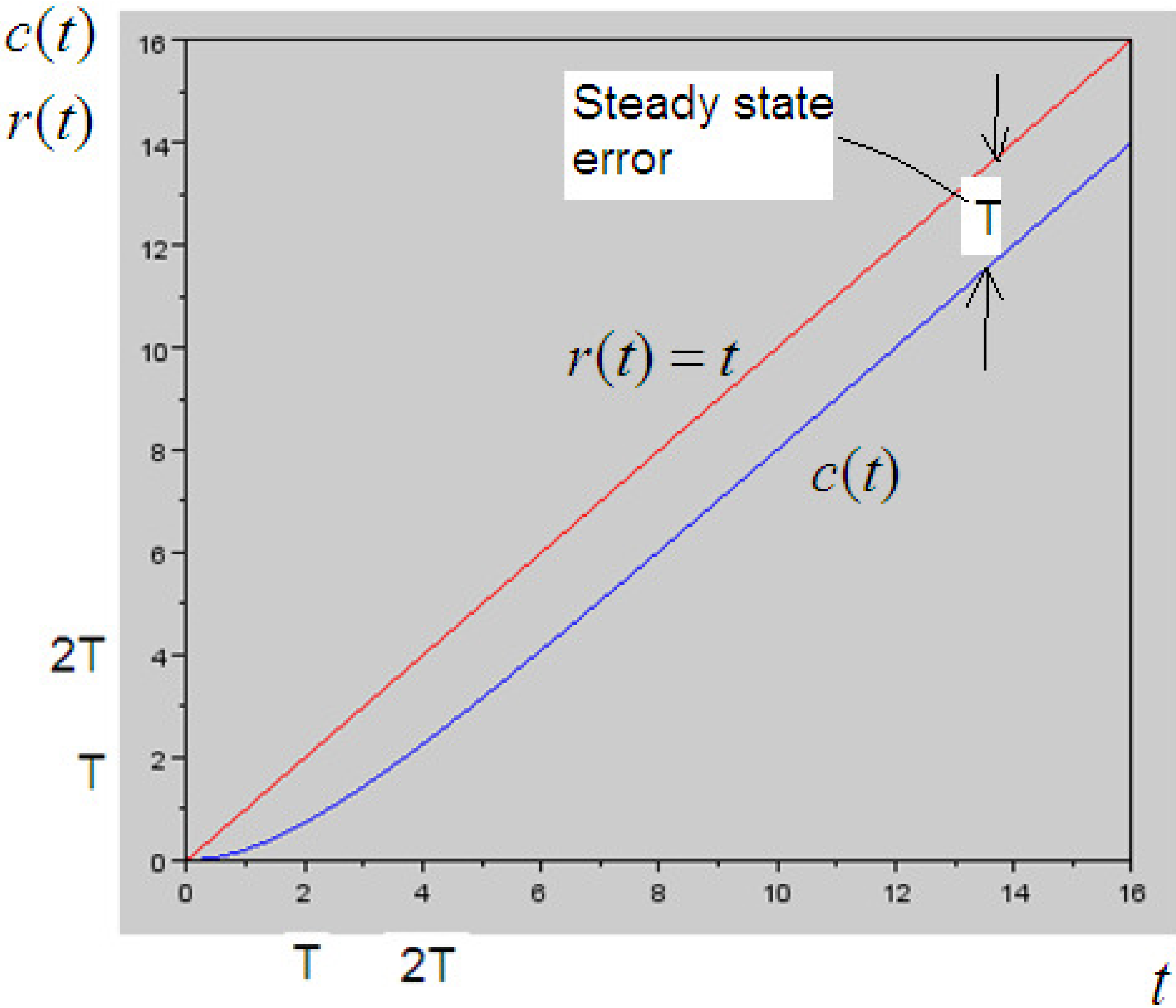
Unit ramp response of first order systems

$$C(s) = G(s)R(s)$$

$$C(s) = \frac{1}{Ts+1} L\{r(t)\}$$

$$= \frac{1}{Ts+1}\frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$c(t) = t - T + Te^{-t/T} \quad \text{for} \quad t >= 0$$



Error signal $e(t) = r(t) - c(t) = T(1 - e^{-t/T})$

At $t = \infty$, $e(\infty) = T$

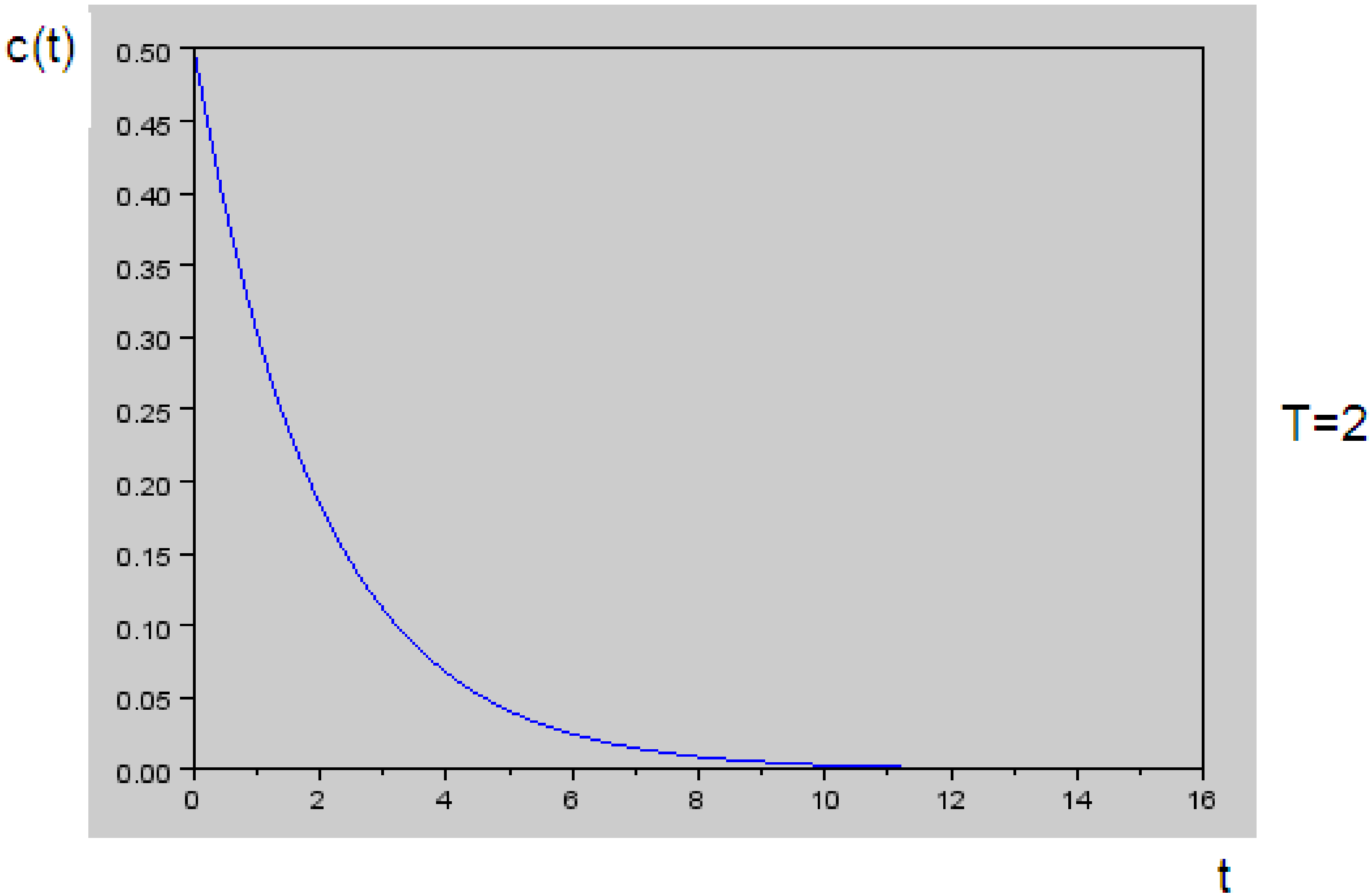
Unit impulse response of first order systems

$$C(s) = G(s)R(s)$$

$$C(s) = \frac{1}{Ts+1} L\{\delta(t)\}$$

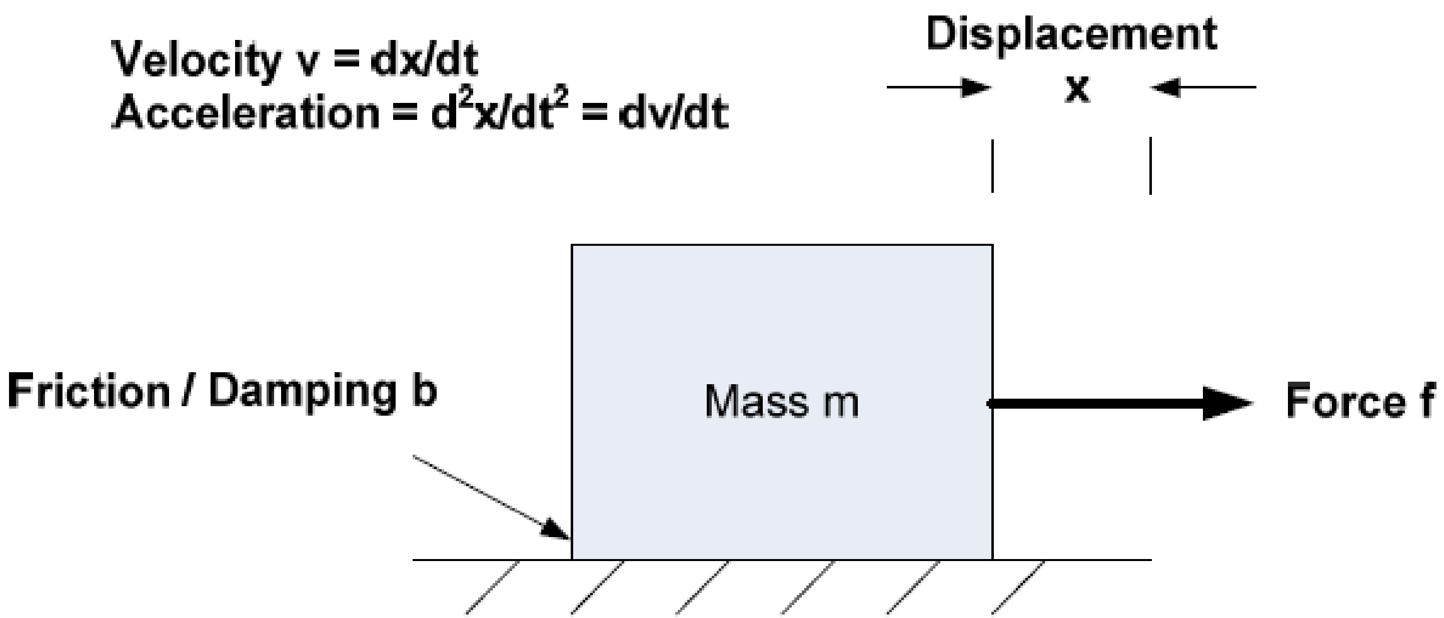
$$= \frac{1}{Ts+1}.1 = \frac{1}{T}\frac{1}{s+1/T}$$

$$c(t) = \frac{1}{T}e^{-t/T} \quad \text{for} \quad t >= 0$$



$$c(t) = \frac{1}{T} e^{-t/T}$$

Example of first order system



First Order Mechanical System

$$G(s)=\frac{v(s)}{F(s)}=\frac{1}{ms+b}=\frac{1/m}{s+b/m}$$

The above relation gives the transfer function of the system in Pole-Zero form with poles at s = -b/m, no finite zero and gain constant = 1/m.

Also, G(s) can be written as,

$$G(s)=\frac{1/b}{(m/b)s+1}$$

τ **= m/b and K = 1/b**

Since F(s) = 1/s for a unit step input, we have

$$V(s) = \frac{1}{s}\frac{K}{\tau s+1}$$

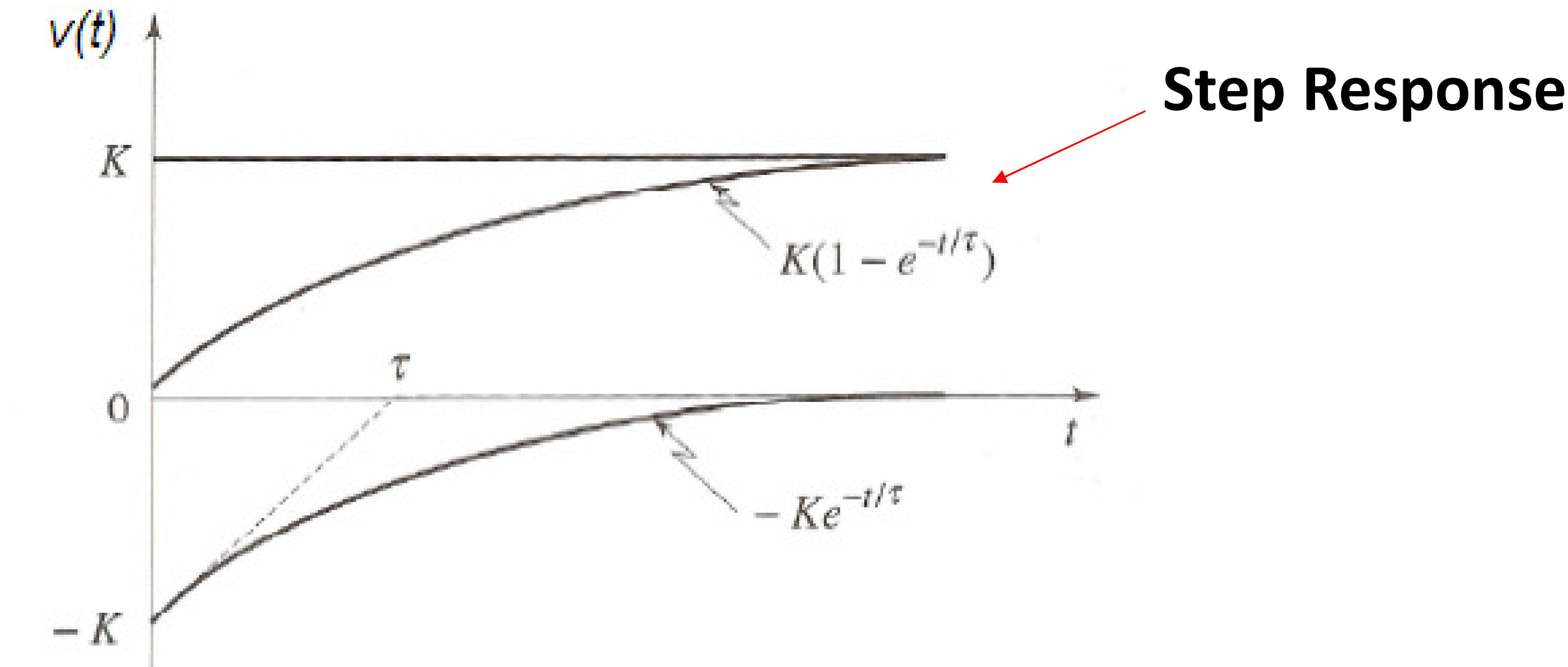
Expanding the above equation by partial fraction we have

$$V(s) = K\left[\frac{1}{s} - \frac{\tau}{s\tau+1}\right]$$

Taking the inverse Laplace Transform we get the unit step response of the first order mechanical (or any other first order system)system as

$$v(t) = K(1 - e^{-t/\tau})$$

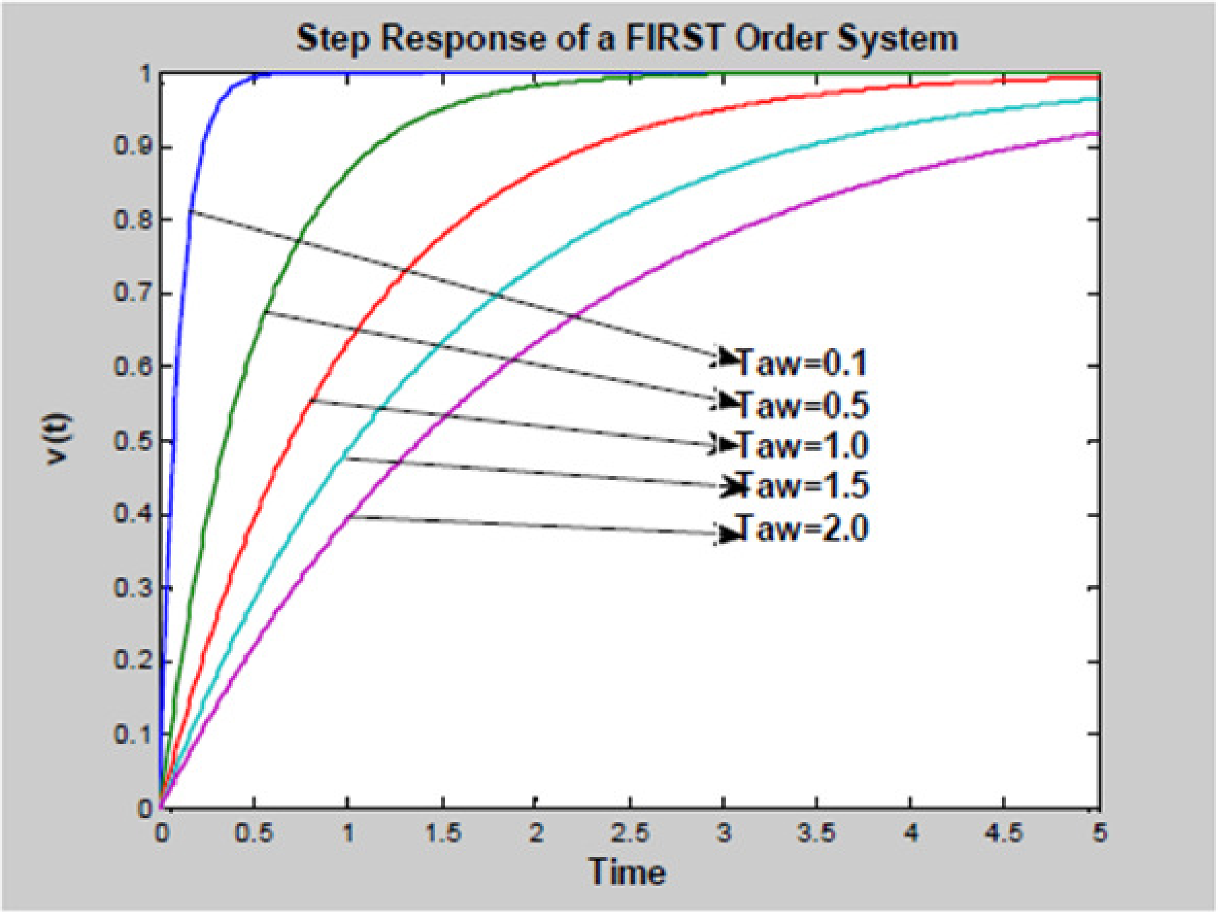
The step response as given by the above Equation is plotted in Figure below; the two components of the response are plotted separately along with the complete response. Mathematically, the exponential term does not decay to zero in a finite length of time. The parameter τ is called the system **time- constant**.



The decay of the exponential term is illustrated with the help of following Table

t	$e^{-t/\tau}$
τ	$e^{-1} = 0.3679$
2τ	$e^{-2} = 0.1353$
3τ	$e^{-3} = 0.0498$
4τ	$e^{-4} = 0.0183$
5τ	$e^{-5} = 0.0067$

For different values of τ, the **step response** of a first order system is also shown in the following Figure.



The parameter K is the ***system gain***, which tells us how much the output variable will change at steady-state in response to a unit change in the input variable.

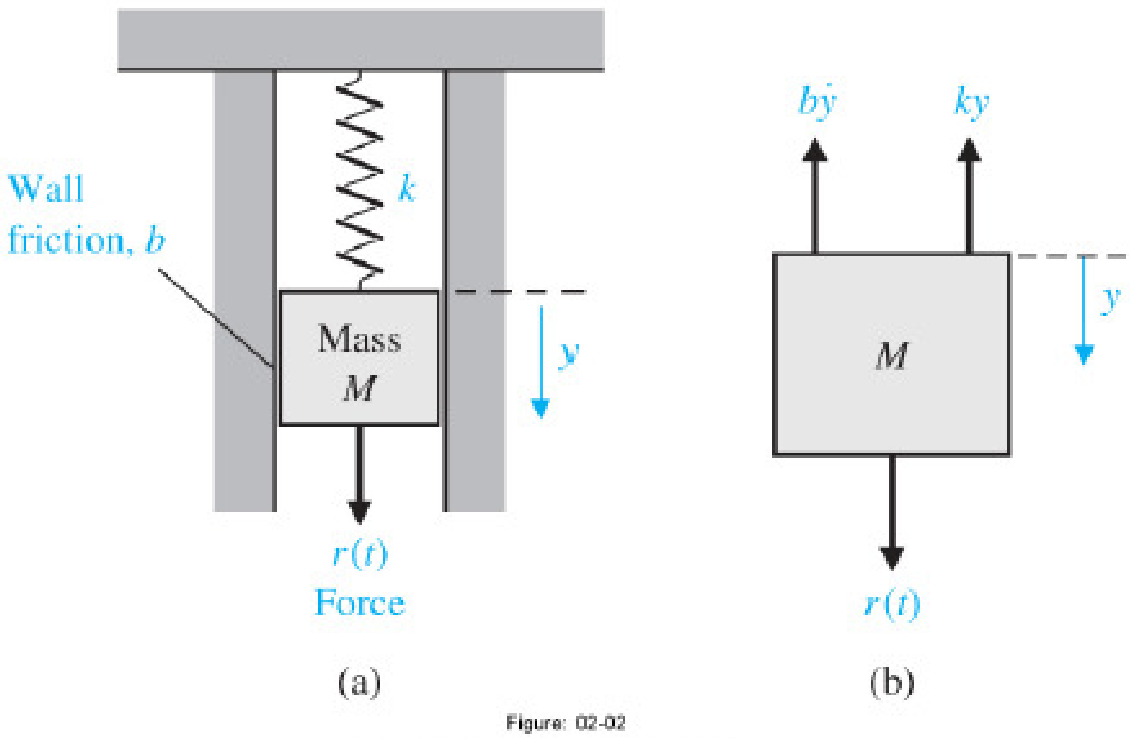
The gain K and time-constant τ are the two parameters which describe the 'personality' of the first-order system. These parameters may be obtained from the physical parameters of the system or experimentally by conducting the step-response test/sinusoidal-response test. The transfer function

$$G(s)=\frac{K}{\tau s+1}$$

is called the **time-constant form** for first-order transfer functions and will be encountered in all types of systems—electrical, mechanical, thermal, hydraulic, etc. A process described by this form of transfer function is called a ***first- order lag*** or **a *simple lag***.

Example of a Second Order System

As an example of a dynamic system represented by a second-order model, we consider here the spring-mass-damper system studied earlier and shown again in the following Figure.



Spring-Mass-Damper System

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

We see that the differential equation is a constant coefficient second order differential equation and hence the system is a second order LTI system.
The transfer function of the system with r(t) as the input and y(t) as the output is

$$G(s) = \frac{Y(s)}{R(s)} = \frac{1}{ms^2 + bs + k} = \frac{1/m}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

The equation gives a second order system with two poles and no finite zero.

The above transfer can be written in a more standard form as

$$G(s) = \frac{Y(s)}{R(s)} = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Where $$K = \frac{1}{k}; \ \omega_n = \sqrt{\frac{k}{m}}; \xi = \frac{1}{2}\frac{b}{\sqrt{km}}$$

ω_n = **Un-damped natural frequency** of the system

ξ = **Damping ratio** of the system

The characteristic equation is

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

The roots of the characteristic equations that are the poles of the transfer function are

$$s_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1} \quad \text{and} \quad s_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

Transfer function of a second order system (K=1)

$$G(s) = \frac{Y(s)}{R(s)} = \frac{\omega_n^{\ 2}}{s^2 + 2\xi\omega_n s + \omega_n^{\ 2}} \qquad (1)$$

ω_n = **Un-damped natural frequency** of the system
ξ = **Damping ratio** of the system

The characteristic equation is

$$s^2 + 2\xi\omega_n s + \omega_n^{\ 2} = 0$$

The roots of the characteristic equations that are the poles of the transfer function are

$$s_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1} \quad \text{and} \quad s_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1}$$

From equation (1), we can write

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} R(s)$$

For unit step

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \frac{1}{s}$$

$$\Rightarrow \quad Y(s) = \frac{-s - 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_n{}^2 \left(\sqrt{1-\xi^2}\right)^2} + \frac{1}{s}$$

$$\Rightarrow \quad y(t) = 1 - \frac{1}{\omega_n \sqrt{1-\xi^2}} \left[(\xi\omega_n)^2 + \omega_n{}^2 \left(\sqrt{1-\xi^2}\right)^2 \right]^{1/2} e^{-\xi\omega_n t} \sin(\omega_n \sqrt{1-\xi^2}\, t + \theta)$$

$$\Rightarrow \quad y(t) = 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin(\omega_n \sqrt{1-\xi^2}\, t + \theta)$$

Assuming $\beta = \sqrt{1\text{-}\xi^2}$

$$y(t) = 1 - \frac{1}{\beta} e^{-\xi\omega_n t} \sin(\omega_n \beta t + \theta)$$

$$\text{where, } \theta = \tan^{-1} \frac{\omega_n \beta}{\xi\omega_n} = \tan^{-1} \frac{\sqrt{1\text{-}\xi^2}}{\xi}$$

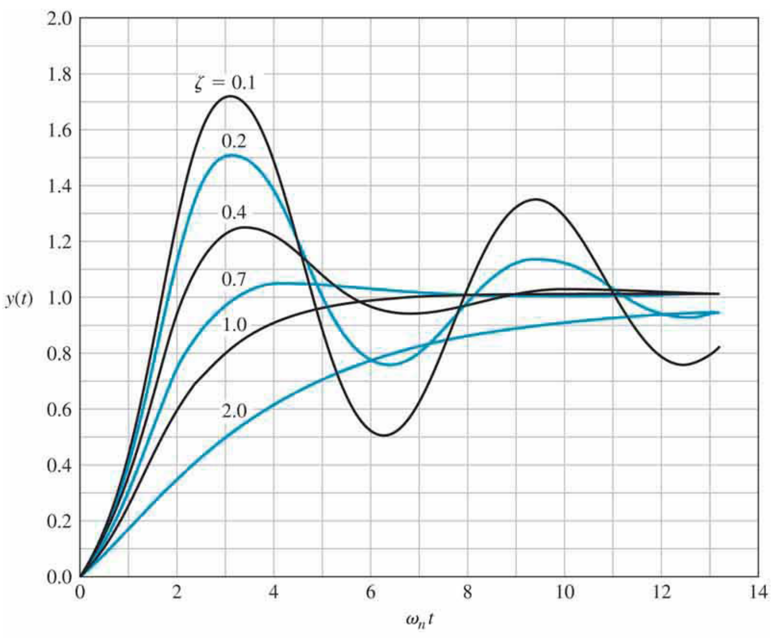
$$\tan\theta = \frac{\sqrt{1\text{-}\xi^2}}{\xi}$$

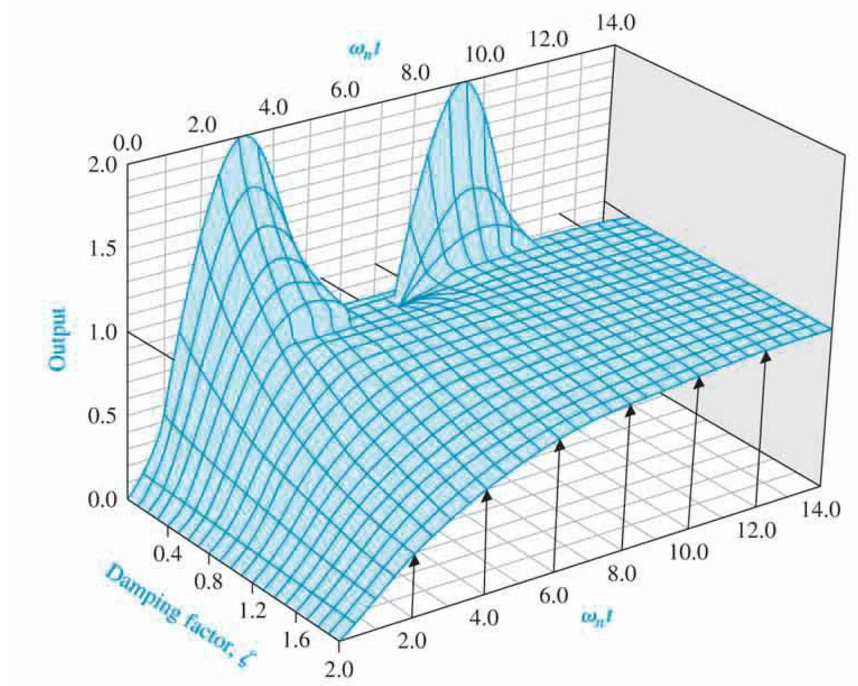
$$\sec^2\theta = 1 + \frac{1-\xi^2}{\xi^2} = \frac{1}{\xi^2}$$

ICs?

$$\theta = \cos^{-1}\xi$$

Step Response of a Second Order System





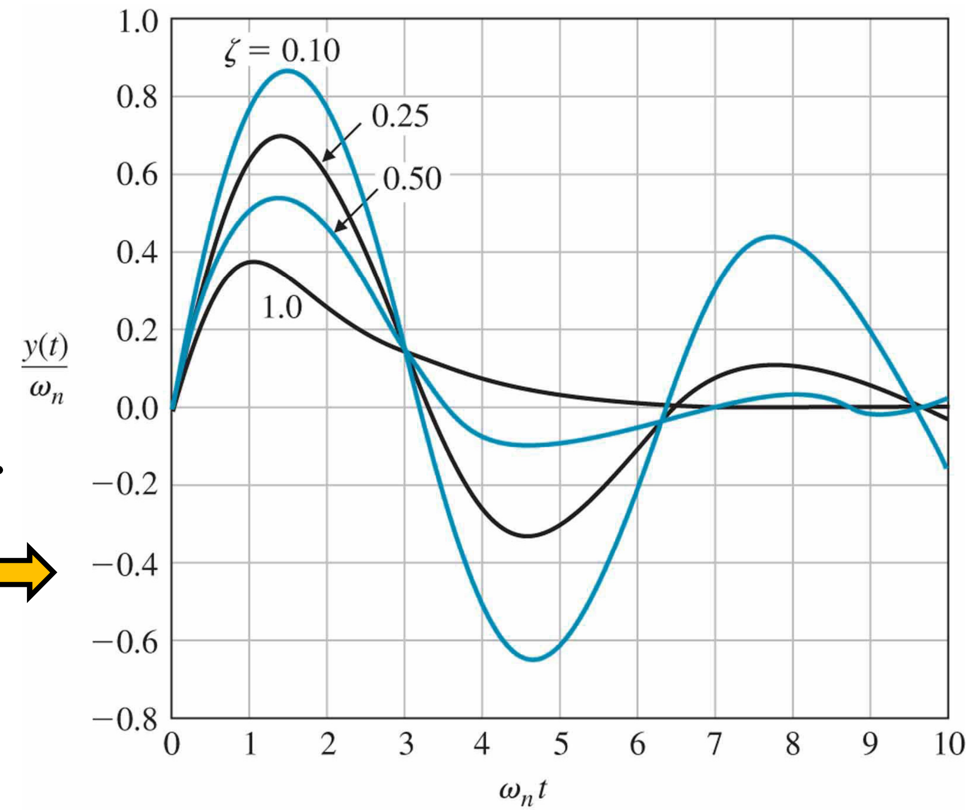
(b)

Unit Impulse Response of a Second Order System

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} R(s)$$

$$= \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} .1$$

$$y(t) = \frac{\omega_n}{\beta} e^{-\xi\omega_n t} \sin(\omega_n \beta t).$$

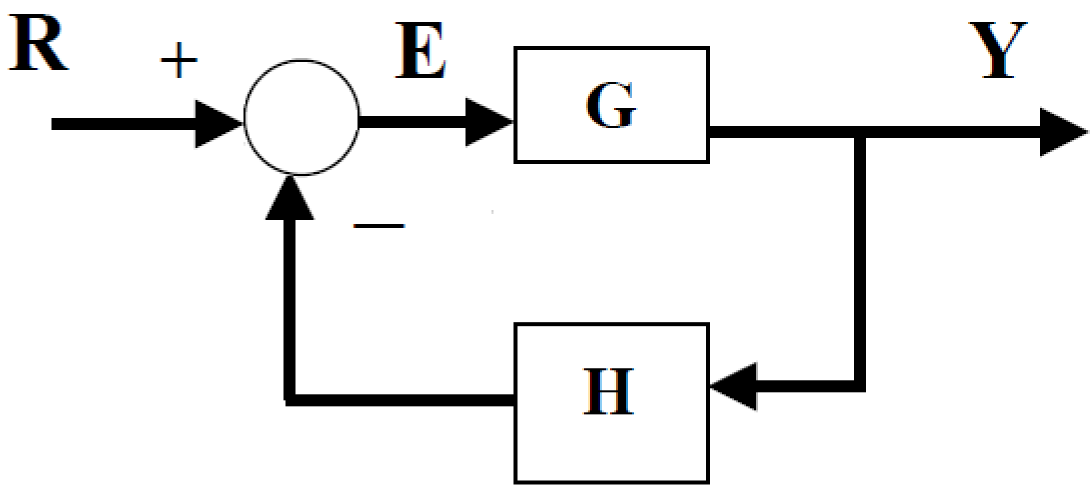


Closed Loop Control

- If G(s)H(s)>>1 for all complex freq. of interest, then:

$$Y(s) = \frac{G}{1+G \cdot H} R \longrightarrow \frac{G}{G \cdot H} R = \frac{1}{H} R$$

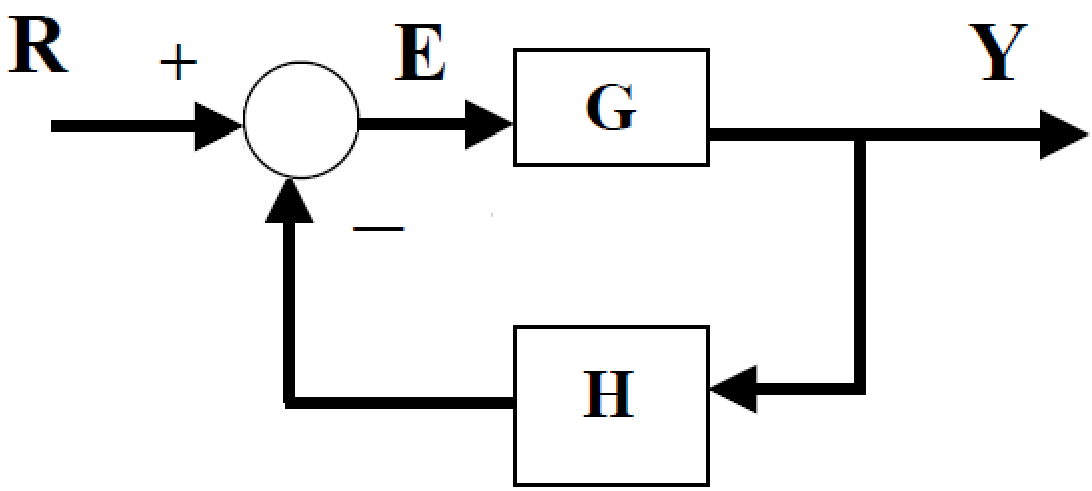
If H = 1, Y(s) = R(s) <u>the desired result</u>



Closed Loop Control

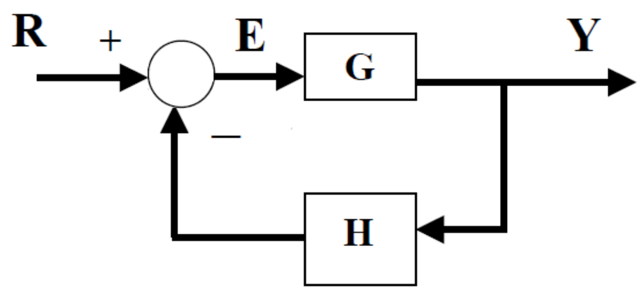
- By increasing the gain of G(s)H(s) it reduces the effect of G(s) on the input -> variation of the parameters of the process, G(s), is reduced *(advantage of a feedback system)*
- But, making G(s)H(s)>>1 can lead to highly oscillatory & even unstable response

$$Y(s) = \frac{G}{1+G\cdot H}R \longrightarrow \frac{G}{G\cdot H}R = \frac{1}{H}R$$



When Process, *G(s)*, is changed

<Open loop> $\Delta Y(s) = \Delta G(s) R(s)$



<Closed loop>

$$Y(s) + \Delta Y(s) = \frac{G(s) + \Delta G(s)}{1 + (G(s) + \Delta G(s))H(s)} R(s)$$

$$Y(s) = \frac{G}{1 + G \cdot H} R$$

Then the change in the output is

$$\Delta Y(s) = \frac{\Delta G(s)}{(1 + GH(s) + \Delta GH(s))(1 + GH(s))} R(s).$$

When GH>>Δ GH(s), as is often the case, we have

$$\Delta Y(s) = \frac{\Delta G(s)}{[1 + GH(s)]^2} R(s).$$

Original T. F.

$$\frac{G}{1 + G \bullet H}$$

Change of the output is reduced by [1 +GH]

System Sensitivity

$$S_G^T = \frac{\text{Ratio of \% change in sys T.F.}}{\text{Ratio of \% change in ``Process'' T.F.}}$$

$$\frac{\Delta T / T}{\Delta G / G} = \frac{\partial \ln T}{\partial \ln G} = \frac{\partial T}{\partial G}\frac{G}{T}$$

<u>Open-loop</u>

$$\Delta Y(s) = \Delta G(s) R(s), \quad \Delta T(s) = \frac{\Delta Y(s)}{R(s)} = \Delta G(s)$$

System Sensitivity

Closed-loop

$$T(s) = \frac{G}{1+GH} \quad \frac{\partial T}{\partial G} = \frac{(1+GH) - G(H)}{(1+GH)^2} = \frac{1}{(1+GH)^2}$$

$$S_G^T = \frac{\partial T}{\partial G} \cdot \frac{G}{T} = \frac{1}{(1+GH)^2} \frac{G}{\frac{G}{1+GH}} = \frac{1}{(1+GH)}$$

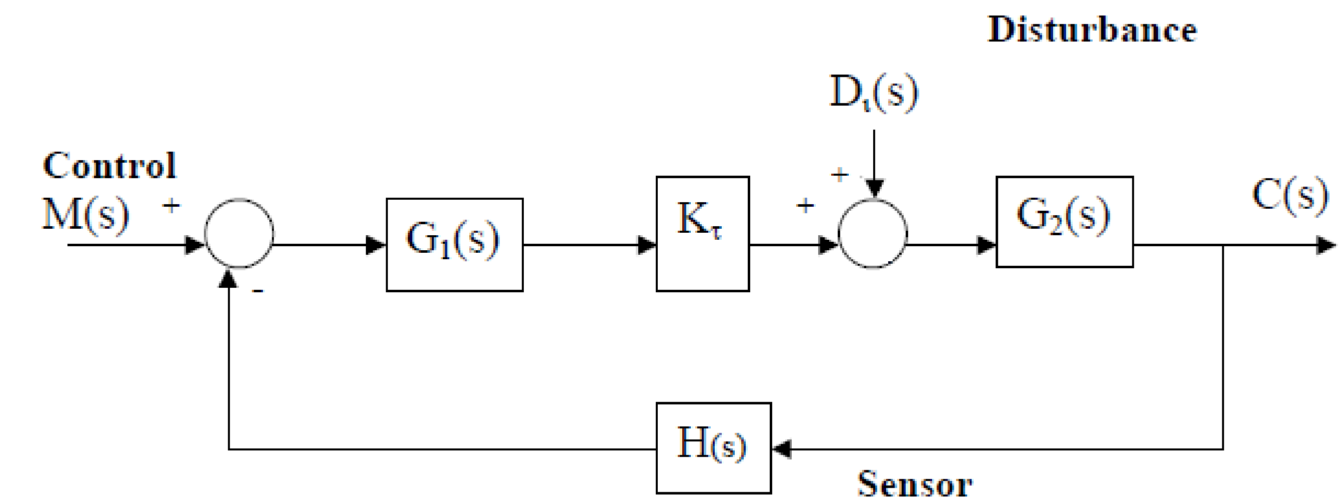
→ Reduced S_G^T below that of the open-loop sys by increasing $G*H$ (>>1.0).

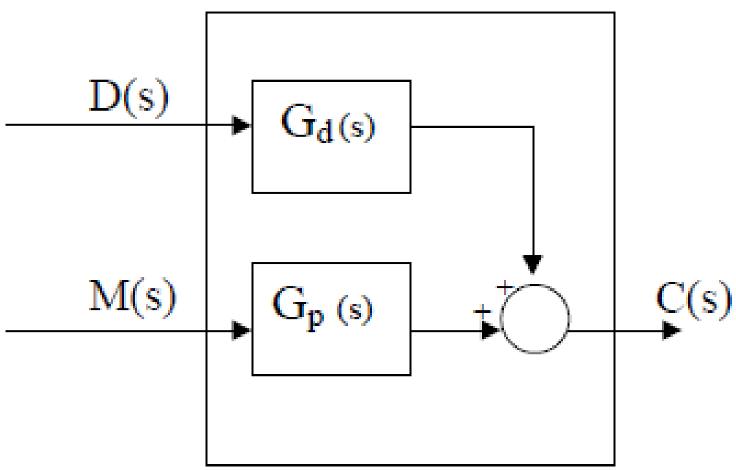
$$S_H^T = \frac{\partial T}{\partial H} \cdot \frac{H}{T} = \frac{-GH}{(1+GH)}$$

* if $GH >> 1.0$ → $S_G^T = -1$

Feedback components should not be varied with environmental changes → change in *H(s)* directly effect output response

Disturbance in a system





By superposition

$$C(s) = \frac{K_\tau G_1 G_2}{1 + K_\tau G_1 G_2 H} M(s) + \frac{G_2}{1 + K_\tau G_1 G_2 H} D\tau(s)$$

$$= G_p(s)M(s) + G_d(s)D_\tau(s)$$

Disturbance in a system

<u>State Variable Model</u>

$$\dot{X} = Ax + Bu \qquad u = \begin{bmatrix} m(t) \\ d(t) \end{bmatrix}$$

$$y = Cx$$

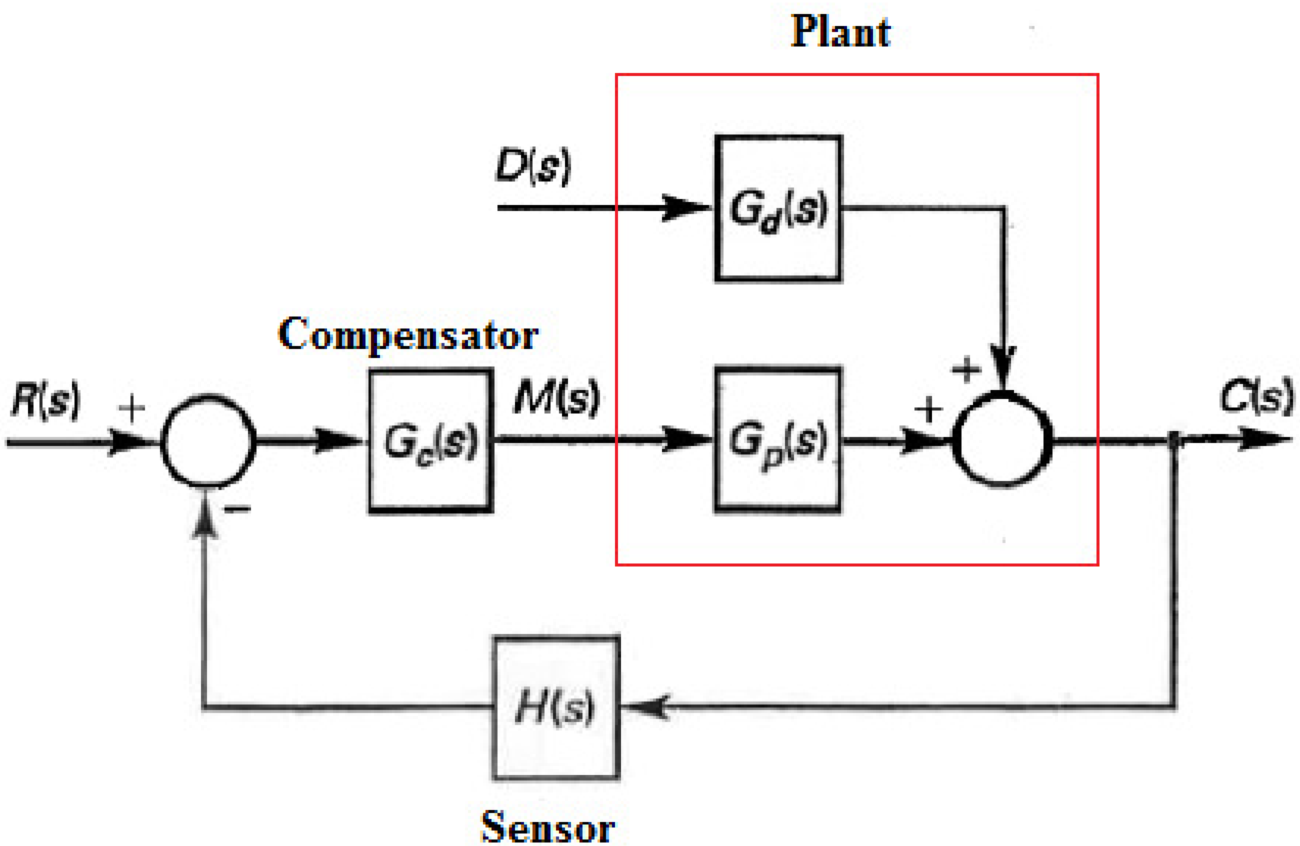
Then T.F. $G(s) = C[sI-A]^{-1}B$

$= [G_p(s) \quad G_d(s)]$

T.F. from the control Input M(s) to the output

T.F. from D(s) to the output

Disturbance in a close loop system



Disturbance in a close loop system

$$C(s) = \left[\frac{G_c G_p}{1 + G_c G_p H}\right] R(s) + \left[\frac{G_d}{1 + G_c G_p H}\right] D(s)$$

$$= T(s)\, R(s) + T_d(s)\, D(s)$$

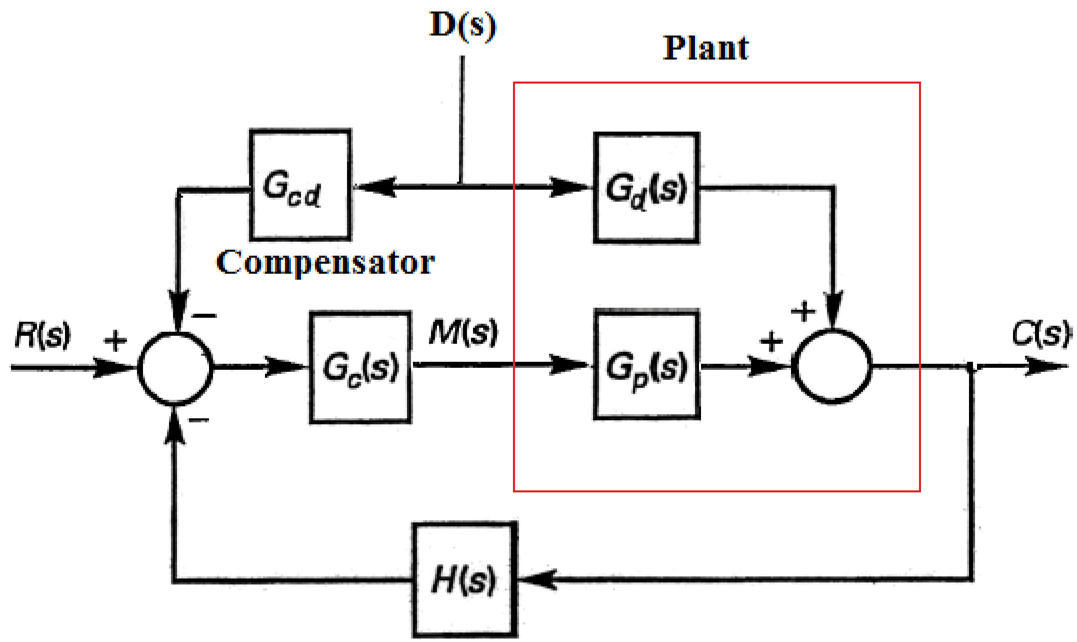
- The loop gain $G_c \bullet G_p \bullet H$ must be made large to reduce the system sensitivity.

$$\longrightarrow \quad T_d = \frac{G_d}{1 + G_c G_p H} = \frac{G_d}{G_c G_p H}$$

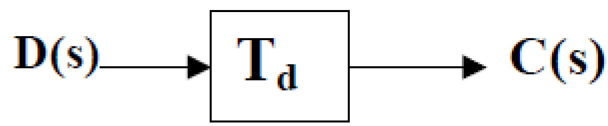
Reducing Disturbance

1. Reduce the gain $G_d(s)$
2. Increase the loop gain $G_c \bullet G_p \bullet H$ (Choice of $\mathbf{G_c}$)
3. Reduce the disturbance ***d(t)***
4. Feed forward method if the disturbance can be measured

Example:
Feed Forward Method



T.F of the disturbance



$$T_d = \frac{G_d}{1+G_cG_PH} + \frac{-G_{cd}G_cG_P}{1+G_cG_PH}$$

if G_{cd} is selected to make $T_d = 0$

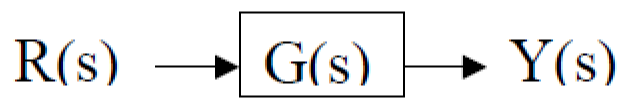
$$G_d - G_{cd}G_cG_P = 0$$

$$\therefore G_{cd} = G_c\frac{G_d}{G_P}$$

Steady State Error

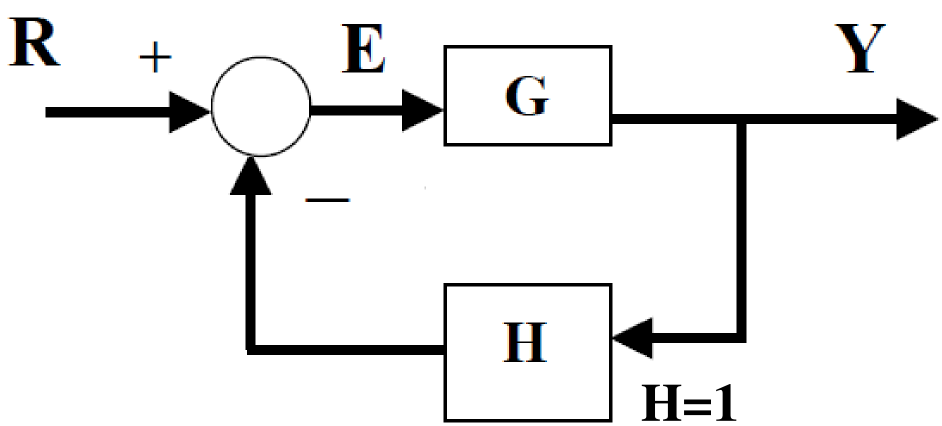
Error after the transient response has decayed (t→inf)

Open Loop



Error $E_o(s) = R(s) - Y(s) = R(s) - G(s)R(s)$
$= R(s)\,[1 - G(s)]$

Closed Loop



Error

$$E_C(s) = \frac{1}{1+G(s)} R(s)$$

Steady State Error

- The final value theorem $\lim_{t\to 0} e(t) = \lim_{s\to 0} sE(s)$
- Error for unit step input

$$u(t) \to \frac{1}{s} \quad t \geq 0$$

1) Open Loop case

$$e_0(\infty) = \lim_{s\to 0} s\left[E(s)\right]\frac{1}{s} = \lim_{s\to 0}\left[1 - G(s)\right] = 1 \text{ - } G(0)$$

2) Closed Loop Case

$$e_c(\infty) = \lim_{s\to 0} s\left[\frac{1}{1+G(s)}\right]\frac{1}{s} = \frac{1}{1+G(0)}$$

→ G(0) is the dc gain and usually greater than 1. If G(0) >> 1, closed-loop error $e_c(\infty)$ is very small

<Example> Consider T.F. $G(s)=\dfrac{K}{s+1}$ Input $\mathrm{R(s)}=\dfrac{1}{s}$

<u>Open Loop case</u>

$$E_c(s)=[1+G(s)]R(s)=[1-G(s)]\frac{1}{s}$$

$$e_0(\infty)=\lim_{s\to 0} E_c(s) \quad =1-G(0)=1-K$$

<u>Closed Loop Case</u>

$$E_c(s)=\frac{1}{1+G(s)}R(s)=\frac{1}{1+G(s)}\frac{1}{s}$$

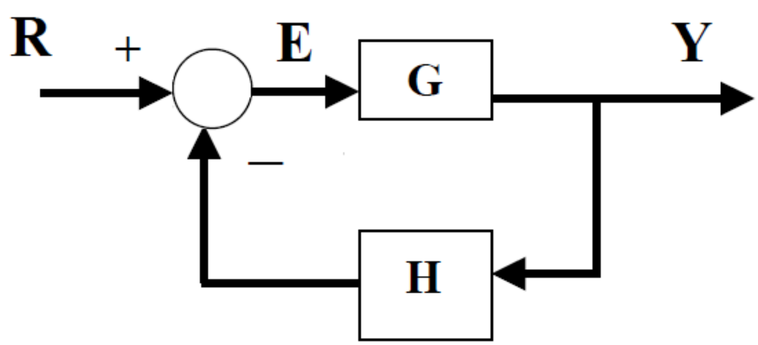
$$e_c(\infty)=\lim_{s\to 0} sE_c(s)=\frac{1}{1+K}$$

<Example> Consider T.F. $G(s) = \frac{K}{s+1}$ Input $R(s) = \frac{1}{s}$

1. For open loop if $K = 1$; $e_0(\infty) = 0$. However, during the operation the parameter of *G(s)* will change due to environment changes.
2. Closed loop error $e_c(\infty)$ can be reduced by selecting high gain of K (if $K = 100$, $e_c(\infty) = \frac{1}{101}$)
3. In case of the gain setting drifts or changes ($\frac{\Delta k}{k} = 0.1$), **open loop** $\Delta e_0(\infty) = 0.1$ (10%) while **closed loop**: $K = 100 \rightarrow 90$

$$\Delta e_c(\infty) = \frac{1}{101} - \frac{1}{91} = 0.0011 \qquad (0.11\%)$$

System Error



$$T = \frac{G}{1-(-GH)} = \frac{G}{1+GH}$$

<u>Output</u> $Y(s) = T(s) \cdot R(s)$

$$Y(s) = \frac{G}{1+GH} R(s)$$

Since $Y(s) = E(s) \cdot G(s) \rightarrow E(s) = Y(s)/G(s)$

$$E(s) = \frac{1}{1+GH} R(s)$$

Actual error = Reference Input – Output ($R - Y$)

System Error

Since $Y(s) = E(s) \cdot G(s) \rightarrow E(s) = Y(s)/G(s)$

$$E(s) = \frac{1}{1 + GH} R(s)$$

Actual error = Reference Input – Output ($R - Y$)

Case 1: $\mathbf{e}_{ss}$ of unity feedback sys. ($\boldsymbol{H} = 1$)

$$E = \frac{1}{1+G} R \quad = R - Y$$

It's the actual error.

Case 2: $\mathbf{e}_{ss}$ of non-unity feedback sys. ($\boldsymbol{H} \neq 1$)

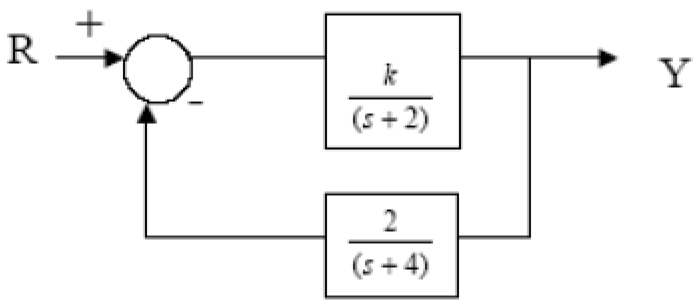
$$E = \frac{1}{1+G \cdot H} R \quad = R - Y \cdot H$$

It's not the actual error.

Use $E = R(s) - Y(s) = R(s) - T(s) \bullet R(s)$

$= [1 - T(s)] \bullet R(s)$

<**Ex**> find ***K*** for a zero steady-state error for $R(s) = \frac{1}{s}$



$$T(s) = \frac{G}{1+GH} = \frac{\frac{K}{(s+2)}}{1+\frac{K}{(s+2)}\frac{2}{(s+4)}} = \frac{K(s+4)}{s^2+6s+8+2K}$$

Using $E(s) = (1 - T)\, R(s)$

$\mathbf{e}_{ss} = lim\ s\,(1 - T)\frac{1}{s} = 1 - T(0) = 1 - \frac{4k}{8+2k} = \frac{8-2k}{8+2k}$

For $\mathbf{e}_{ss} = 0 \rightarrow \quad 8 - 2k = 0 \rightarrow \boldsymbol{K} = 4$