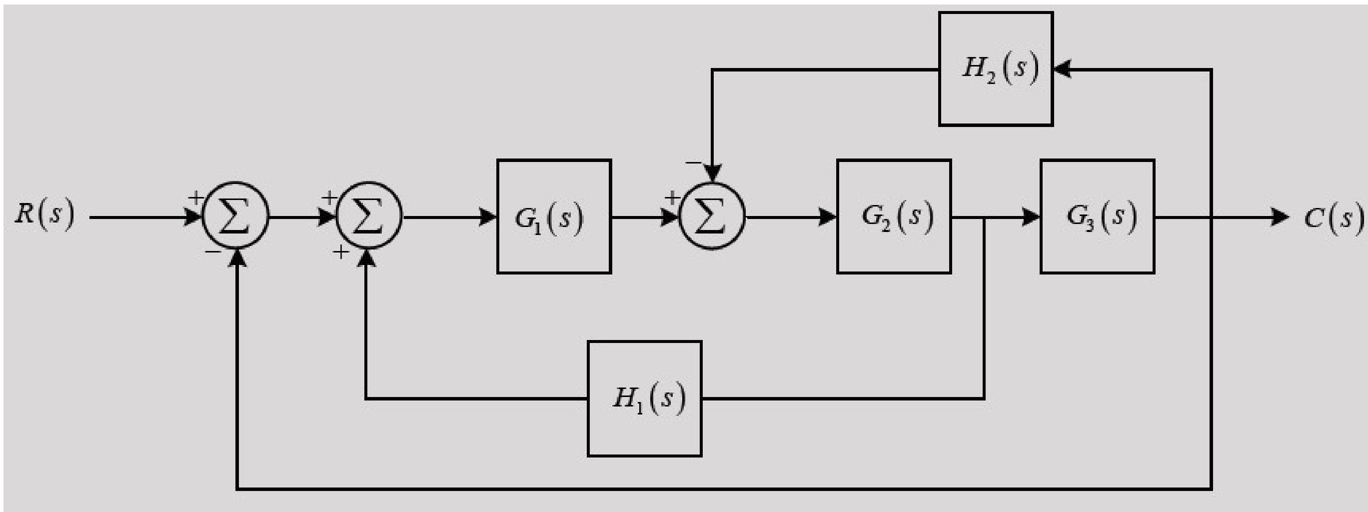
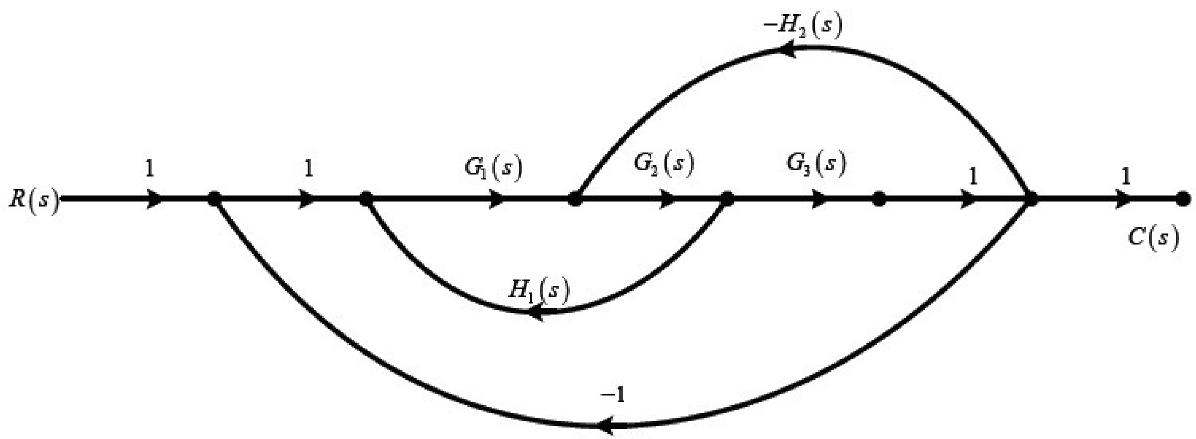
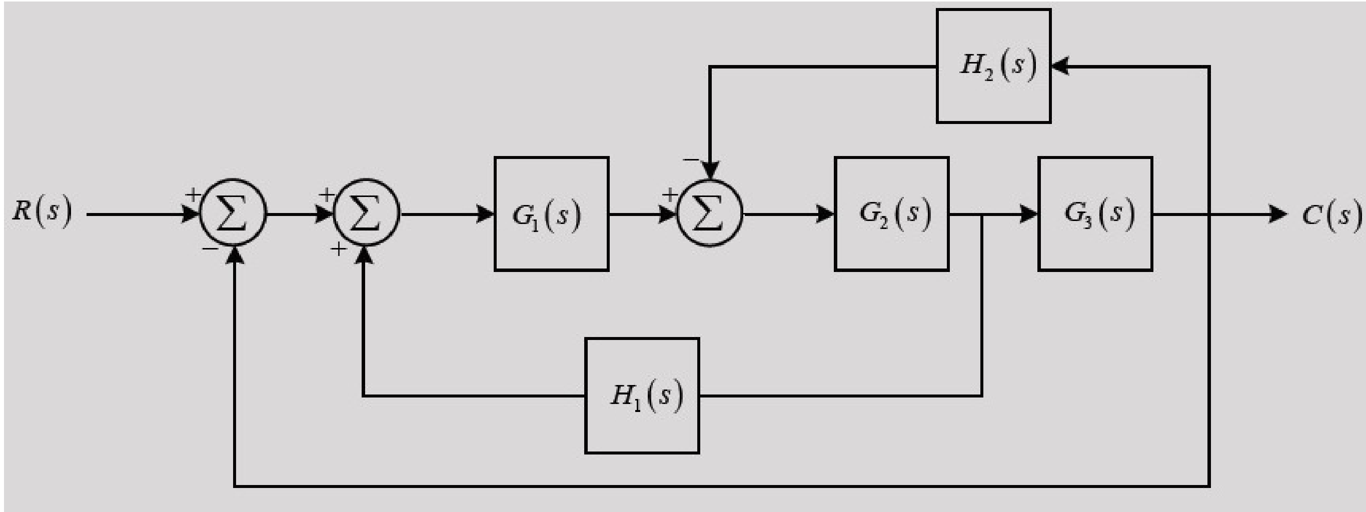
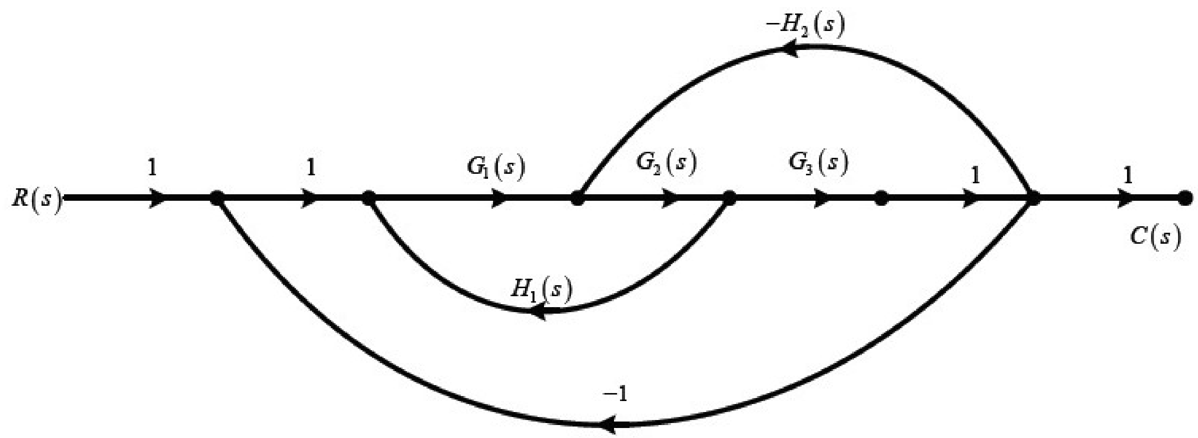
Example 1:
Transfer function of the system







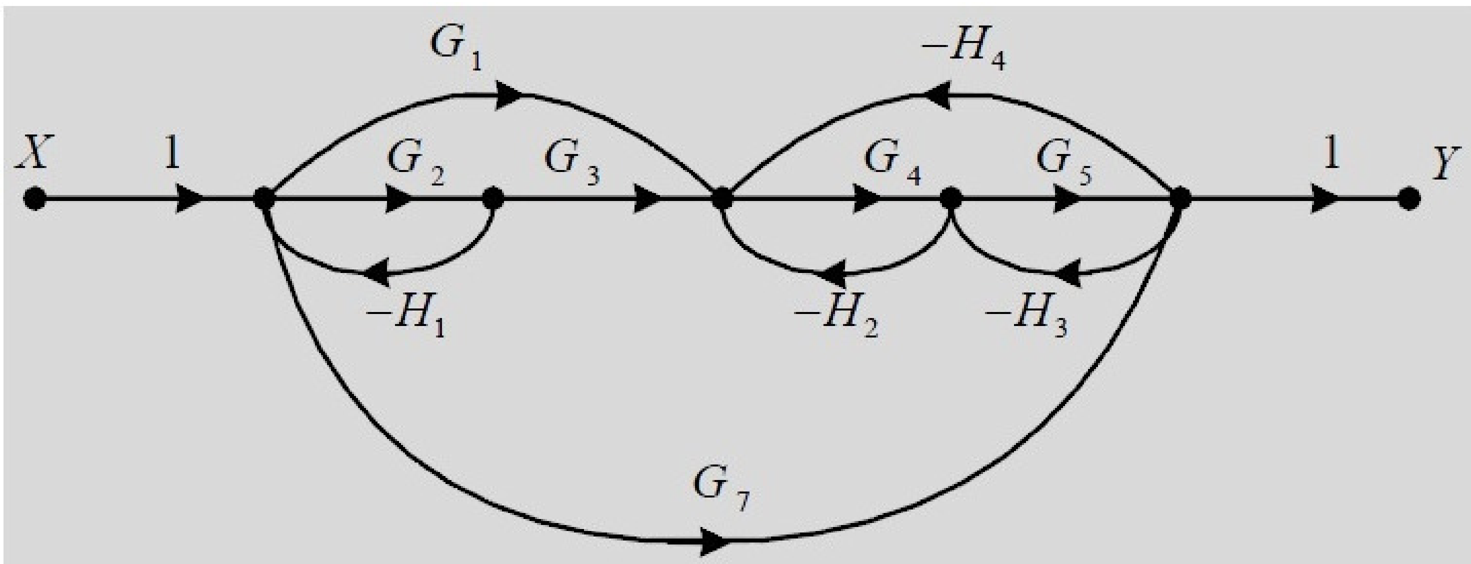


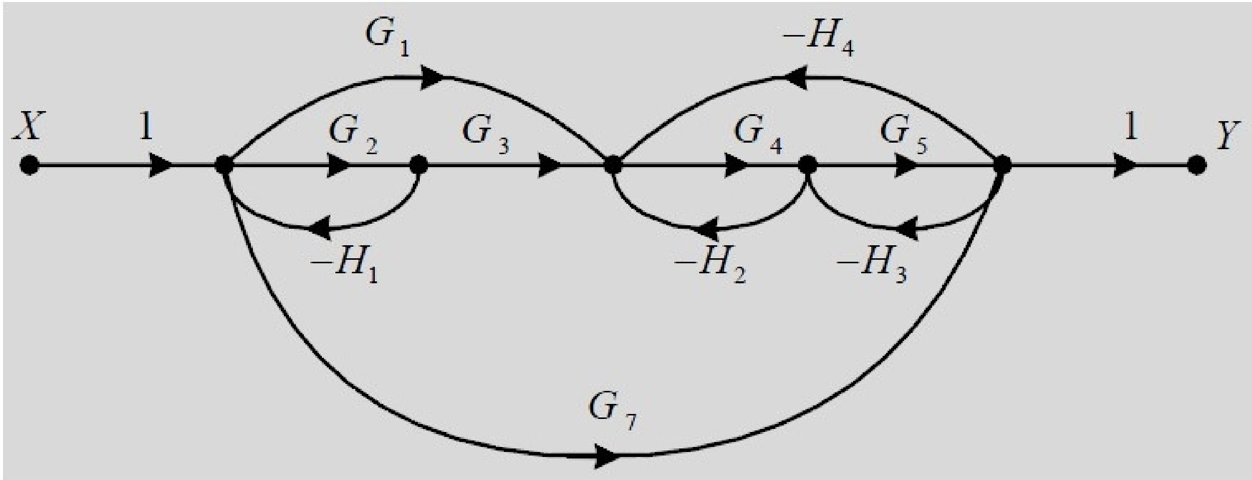
$$\Delta_1 = 1 - 0 = 1$$

$$\Delta = 1 - \left(G_1(s)G_2(s)H_1(s) - G_2(s)G_3(s)H_2(s) - G_1(s)G_2(s)G_3(s)\right) + (0) - \ldots$$

$$M(s) = \frac{G_1(s)G_2(s)G_3(s)}{1 - G_1(s)G_2(s)H_1(s) + G_2(s)G_3(s)H_2(s) + G_1(s)G_2(s)G_3(s)}$$

Example 2:
Transfer function of the system





$$j=1 \rightarrow M_1 = 1 \times G_2 \times G_3 \times G_4 \times G_5 \times 1$$
$$j=2 \rightarrow M_2 = 1 \times G_1 \times G_4 \times G_5 \times 1$$
$$j=3 \rightarrow M_3 = 1 \times G_7 \times 1$$

$$\Delta_1 = 1 - 0 = 1$$
$$\Delta_2 = 1 - 0 = 1$$
$$\Delta_3 = 1 - \left(-G_4 H_2\right) = 1 + G_4 H_2$$

$$\begin{aligned}\Delta = 1 &- \left(-G_2 H_1 - G_4 H_2 - G_5 H_3 - G_4 G_5 H_4\right) \\ &+ \left(\left(-G_2 H_1\right)\left(-G_4 H_2\right) + \left(-G_2 H_1\right)\left(-G_5 H_3\right) + \left(-G_2 H_1\right)\left(-G_4 G_5 H_4\right)\right) = \\ &= 1 + G_2 H_1 + G_4 H_2 + G_5 H_3 + G_4 G_5 H_4 + G_2 H_1 G_4 H_2 + G_2 H_1 G_5 H_3 + G_2 H_1 G_4 G_5 H_4\end{aligned}$$

$$M(s) = \frac{G_2 G_3 G_4 G_5 + G_1 G_4 G_5 + G_7\left(1 + G_4 H_2\right)}{1 + G_2 H_1 + G_4 H_2 + G_5 H_3 + G_4 G_5 H_4 + G_2 H_1 G_4 H_2 + G_2 H_1 G_5 H_3 + G_2 H_1 G_4 G_5 H_4}$$

Chapter 3

State Variable Models

The State Variables of a Dynamic System

The State Differential Equation

Signal-Flow Graph State Variables

The Transfer Function from the State Equation

Introduction

- In the previous chapter, we used Laplace transform to obtain the transfer function models representing linear, time-invariant, physical systems utilizing block diagrams to interconnect systems.
- In Chapter 3, we turn to an alternative method of system modeling using **time-domain methods**.
- In Chapter 3, we will consider physical systems described by an **nth-order ordinary differential equations**.
- Utilizing a set of variables known as **state variables**, we can obtain a set of first-order differential equations.
- The time-domain state variable model lends itself easily to computer solution and analysis.

Time-Varying Control System

- With the ready availability of digital computers, it is convenient to consider the time-domain formulation of the equations representing control systems.

- The time-domain is the mathematical domain that incorporates the response and description of a system in terms of time t.

- The time-domain techniques can be utilized for nonlinear, time-varying, and multivariable systems (a system with several input and output signals).

- A time-varying control system is a system for which one or more of the parameters of the system may vary as a function of time.

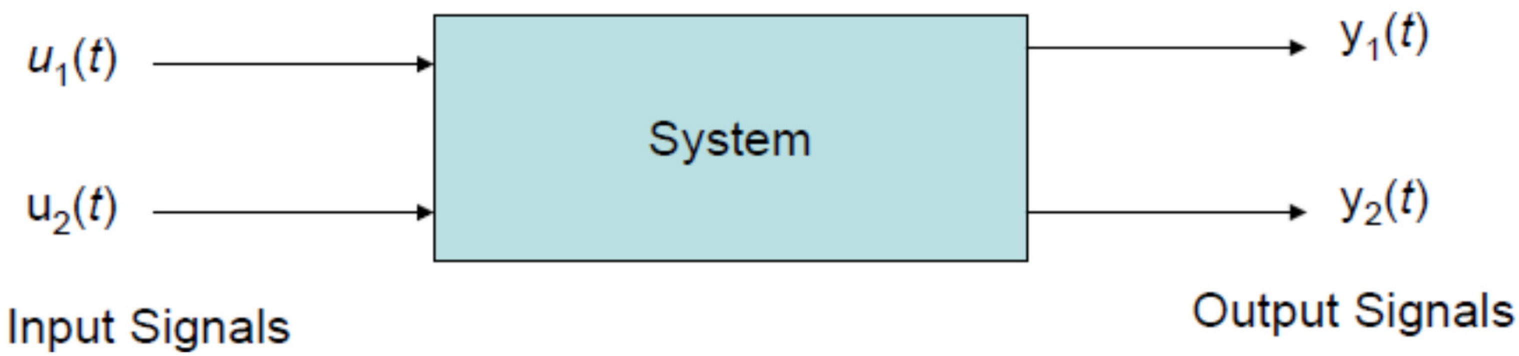
- For example, the mass of a missile varies as a function of time as the fuel is expended during flight

Terms

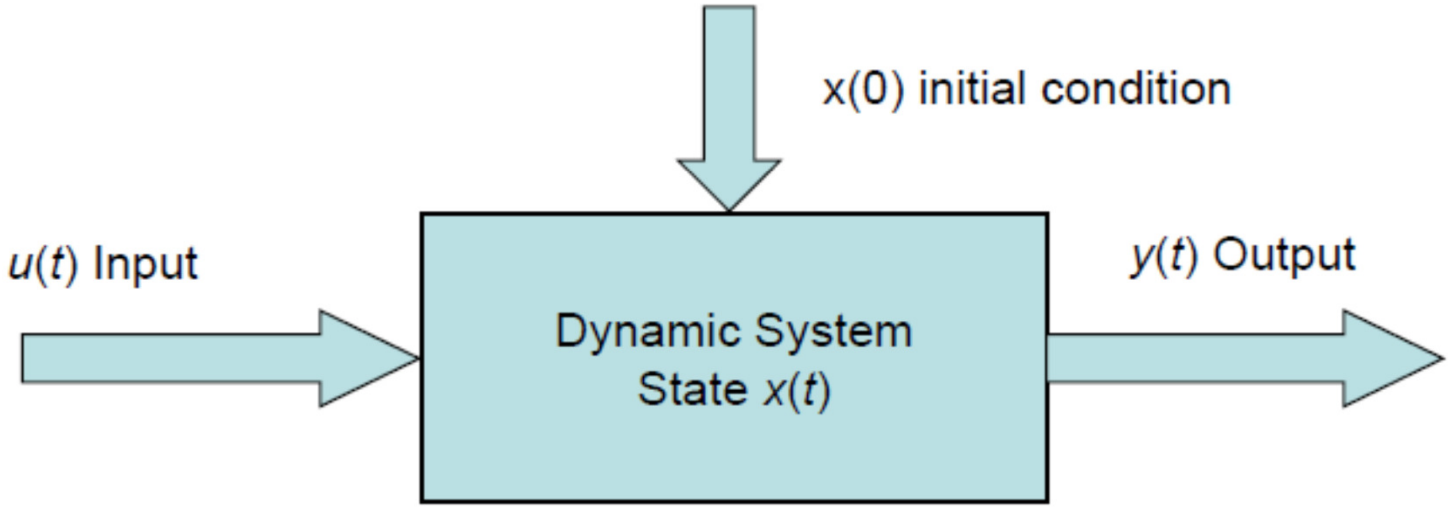
- **State:** The state of a dynamic system is the smallest set of variables (called state variables) so that the knowledge of these variables at $t = t_0$, together with the knowledge of the input for $t \geq t_0$, determines the behavior of the system for any time $t \geq t_0$.
- **State Variables:** The state variables of a dynamic system are the variables making up the smallest set of variables that determine the state of the dynamic system.
- **State Vector:** If n state variables are needed to describe the behavior of a given system, then the n state variables can be considered the n components of a vector x. Such vector is called a state vector.
- **State Space:** The n-dimensional space whose coordinates axes consist of the x_1 axis, x_2 axis, .., x_n axis, where x_1, x_2, .., x_n are state variables, is called a state space.
- **State-Space Equations:** In state-space analysis, we are concerned with three types of variables that are involved in the modeling of dynamic system: input variables, output variables, and state variables.

The State Variables of a Dynamic System

- The state of a system is a set of variables such that the knowledge of these variables and the input functions will, with the equations describing the dynamics, provide the future state and output of the system.
- For a dynamic system, the state of a system is described in terms of a set of state variables.



State Variables of a Dynamic System



The state variables describe the future response of a system, given the present state, the excitation inputs, and the equations describing the dynamics

The State Differential Equation

The state of a system is described by the set of first-order differential equations written in terms of the state variables (x_1, x_2, .., x_n)

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + \ldots + b_{2m}u_m$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nm}u_m$$

$$\dot{x} = \frac{dx}{dt}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ . & . & . \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} \ldots b_{1m} \\ \ldots\ldots\ldots \\ b_{n1} \ldots b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ . \\ u_m \end{bmatrix}$$

A x B u

A : State matrix; B : input matrix

C : Output matrix; D : direct transmission matrix

$$\dot{x} = Ax + Bu \text{ (State differential equation)}$$

$$y = Cx + Du \text{ (Output equation - output signals)}$$

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = w(t)$$

$$x_1(t) = y(t)$$

$$x_2(t) = \dot{x}_1(t) = \frac{dy(t)}{dt}$$

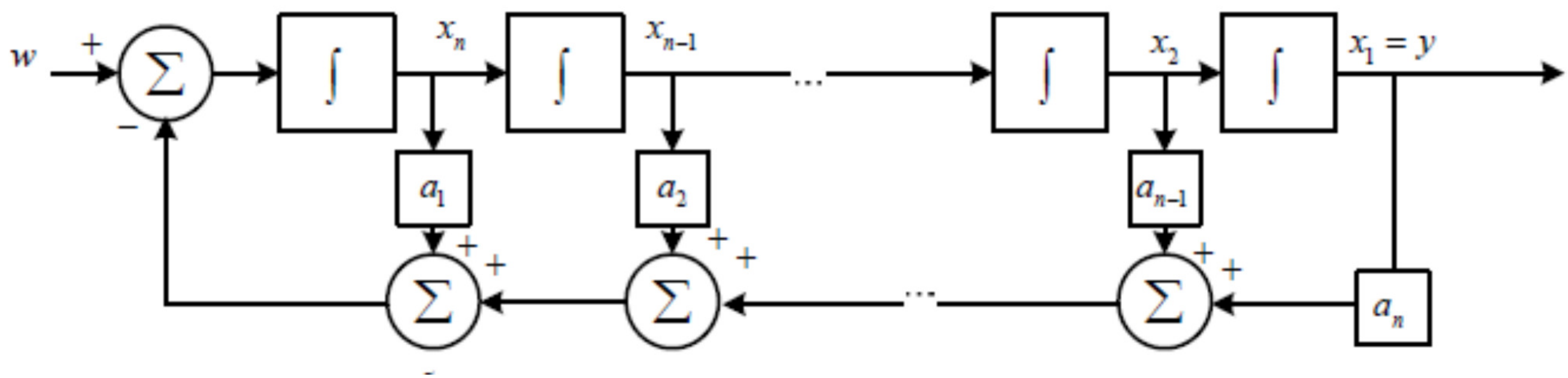
$$\vdots$$

$$x_{n-1}(t) = \dot{x}_{n-2}(t) = \frac{d^{n-2} y(t)}{dt^{n-2}}$$

$$x_n(t) = \frac{d^{n-1} y(t)}{dt^{n-1}}$$

$$\rightarrow \dot{x}_n(t) = \frac{d^n y(t)}{dt^n} = \left(-a_1 \underbrace{\frac{d^{n-1} y(t)}{dt^{n-1}}}_{x_n(t)} - \ldots - a_{n-1} \underbrace{\frac{dy(t)}{dt}}_{x_2(t)} - a_n \underbrace{y(t)}_{x_1(t)} + w(t) \right)$$

$$= -a_1 x_n(t) - \ldots - a_{n-1} x_2(t) - a_n x_1(t) + w(t)$$



w
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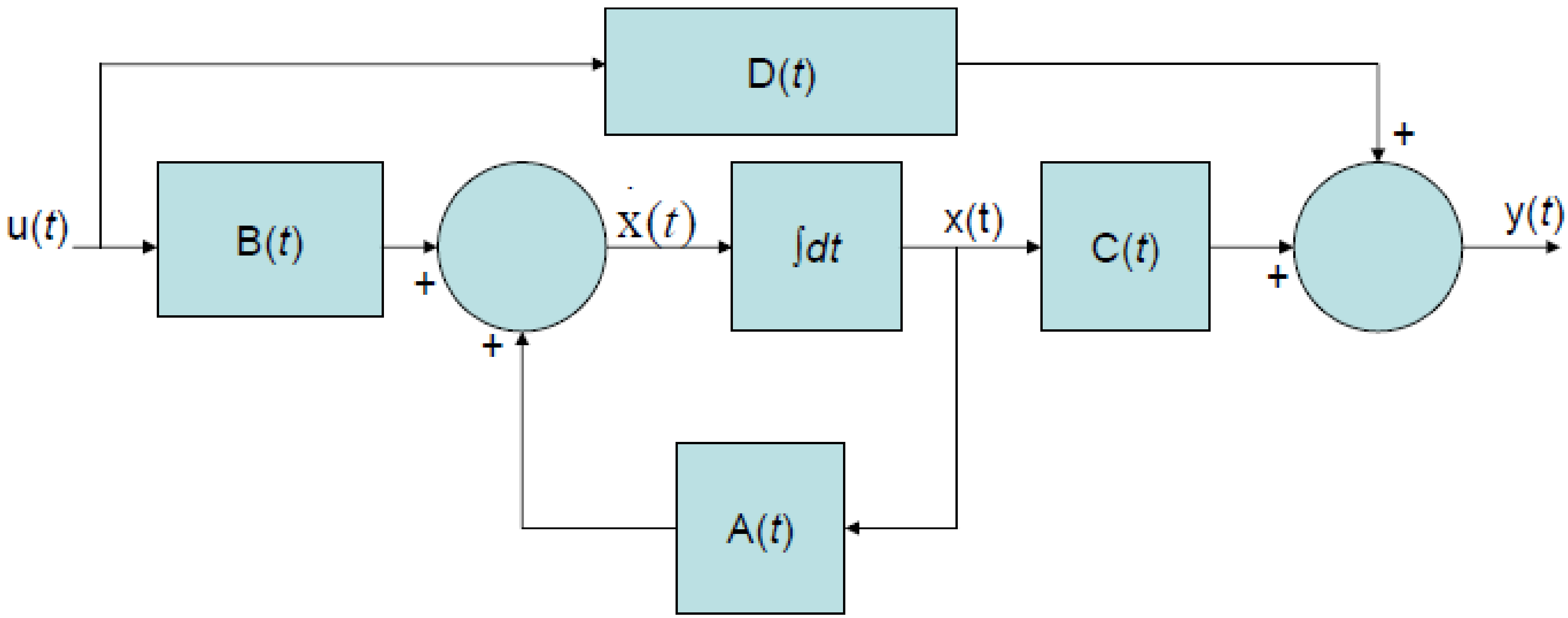
x_n
x_{n-1}
x_2
$x_1 = y$
a_1
a_2
a_{n-1}
a_n

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} w \\ y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + 0 \times w \end{cases}$$

Block Diagram of the Linear, Continuous Time Control System



$$\dot{x}(t) = A(t)x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

Mechanical Example: Mass-Spring Damper

A set of state variables sufficient to describe this system includes the position and the velocity of the mass, therefore, we will define a set of state variables as (x_1, x_2)

$$x_1(t) = y(t)$$

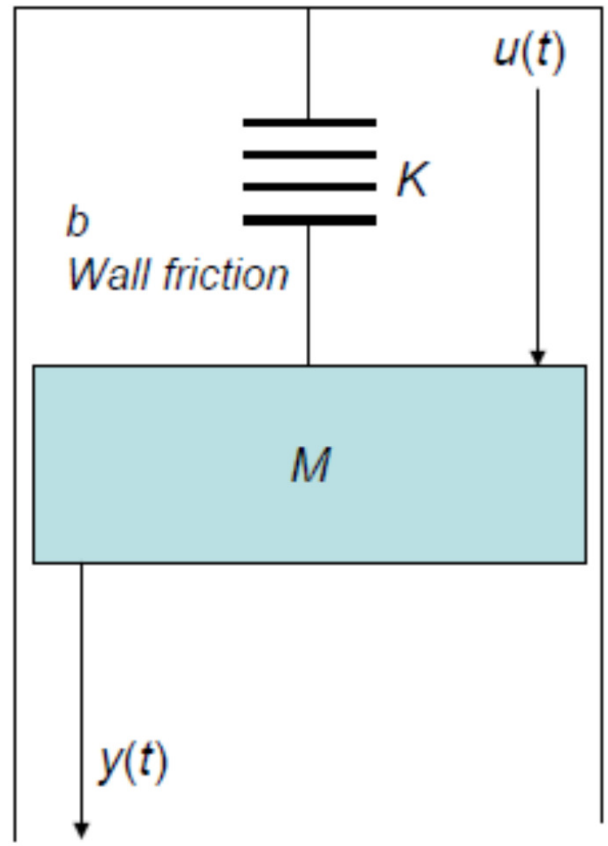
$$x_2(t) = \frac{dy(t)}{dt}$$

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$$

$$M\frac{dx_2}{dt} + bx_2 + kx_1 = u(t)$$

$$\frac{dx_1}{dt} = x_2;$$

$$\frac{dx_2}{dt} = -\frac{b}{m}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u$$



k : Spring constant

$$m\ddot{y}+b\dot{y}+ky=u$$

This is a second order system. It means it involves two integrators.

Let us define two variables: $x_1(t)$ and $x_2(t)$

$$x_1(t)=y(t);\ x_2(t)=\dot{y}(t);\ \text{then } \dot{x}_1=x_2$$

$$\dot{x}_2=-\frac{k}{m}x_1-\frac{b}{m}x_2+\frac{1}{m}u$$

The output equation is: $y=x_1$

In a vector matrix form, we have

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x1\\ x2\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}u \text{ (State Equation)}$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \text{ (Output Equation)}$$

The state equation and the output equation are in the standard form:

The Transfer Function from the State Equation

Given the transfer function $G(s)$, we may obtain the state variable equations using the signal-flow graph model. Recall the two basic equations

$$\dot{\mathrm{x}} = \mathrm{A}\mathrm{x} + \mathrm{B}u$$

$$y = \mathrm{C}\mathrm{x}$$

y is the single output and u is the single input.

Take the Laplace transform

$$s\mathrm{X}(s) = \mathrm{A}\,\mathrm{X}(s) + \mathrm{B}\,U(s)$$

$$Y(s) = \mathrm{C}\mathrm{X}(s)$$

$$(s\mathrm{I} - \mathrm{A})\mathrm{X}(s) = \mathrm{B}U(s)$$

$$\text{Since } [s\mathrm{I} - \mathrm{A}]^{-1} = \Phi(s)$$

$$\mathrm{X}(s) = \Phi(s)\,\mathrm{B}\,U(s)$$

$$Y(s) = \mathrm{C}\,\Phi(s)\,\mathrm{B}\,U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \mathrm{C}\,\Phi(s)\,\mathrm{B}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}2\\ 2\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+0u$$

$$A=\begin{bmatrix}-2 & 1\\ 1 & -4\end{bmatrix}, B=\begin{bmatrix}2\\ 2\end{bmatrix}, C=\begin{bmatrix}1 & 1\end{bmatrix}, D=0$$

$$G(s)=C(sI-A)^{-1}B+D$$

$$\rightarrow (sI-A)^{-1}=\left(s\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}-2 & 1\\ 1 & -4\end{pmatrix}\right)^{-1}=\begin{pmatrix}s+2 & -1\\ -1 & s+4\end{pmatrix}^{-1}$$

$$=\frac{1}{(s+2)(s+4)-1}\begin{pmatrix}s+4 & 1\\ 1 & s+2\end{pmatrix}=\frac{1}{s^2+6s+7}\begin{pmatrix}s+4 & 1\\ 1 & s+2\end{pmatrix}$$

$$\rightarrow G(s)=\begin{pmatrix}1 & 1\end{pmatrix}\left(\frac{1}{s^2+6s+7}\begin{pmatrix}s+4 & 1\\ 1 & s+2\end{pmatrix}\right)\begin{pmatrix}2\\ 2\end{pmatrix}=\frac{1}{s^2+6s+7}\left(\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}s+4 & 1\\ 1 & s+2\end{pmatrix}\begin{pmatrix}2\\ 2\end{pmatrix}\right)$$

$$=\frac{1}{s^2+6s+7}\left(\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}2s+10\\ 2s+6\end{pmatrix}\right)=\frac{4s+16}{s^2+6s+7}$$

$$G(s) = \frac{4s+16}{s^2+6s+7} \rightarrow Y(s) = \frac{4s+16}{s\left(s^2+6s+7\right)} = \frac{2.28}{s} - \frac{2.15}{s+1.58} - \frac{0.13}{s+4.41}$$

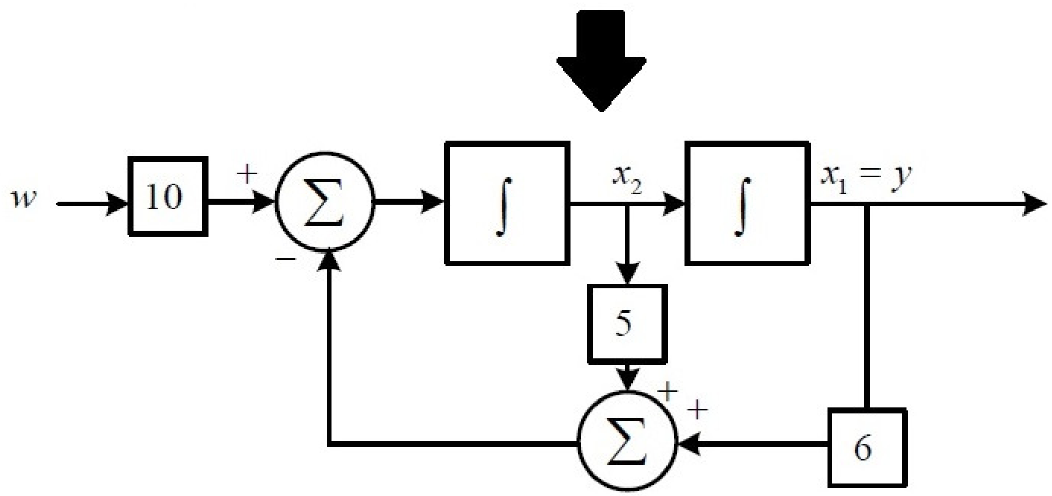
$$\rightarrow y(t) = \left(2.28 - 2.15e^{-1.58t} - 0.13e^{-4.41t}\right)u(t)$$

$$G(s)=\frac{10}{s^2+5s+6}$$

$$\begin{cases} x_1(t)=y(t) \\ \\ x_2(t)=\dot{x}_1(t)=\dfrac{dy(t)}{dt} \\ \dot{x}_2(t)=\dfrac{d^2y(t)}{dt^2}=-5\underbrace{\dfrac{dy(t)}{dt}}_{x_2(t)}-6\underbrace{y(t)}_{x_1(t)}+10\,w(t) \end{cases}$$

$$G(s)=\frac{10}{s^2+5s+6}$$

$$\begin{cases} x_1(t)=y(t) \\ x_2(t)=\dot{x}_1(t)=\dfrac{dy(t)}{dt} \\ \dot{x}_2(t)=\dfrac{d^2y(t)}{dt^2}=-5\underbrace{\dfrac{dy(t)}{dt}}_{x_2(t)}-6\underbrace{y(t)}_{x_1(t)}+10\,w(t) \end{cases}$$



$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = b_0 \frac{d^n w(t)}{dt^n} + b_1 \frac{d^{n-1} w(t)}{dt^{n-1}} + \ldots + b_{n-1} \frac{dw(t)}{dt} + b_n w(t)$$

$$x_1(t) = y(t) - \beta_0 w(t)$$

$$x_2(t) = \dot{x}_1(t) - \beta_1 w(t)$$

$$\vdots$$

$$x_n(t) = \dot{x}_{n-1}(t) - \beta_{n-1} w(t)$$

$$\rightarrow \dot{x}_n(t) = -a_1 x_n(t) - \ldots - a_{n-1} x_2(t) - a_n x_1(t) + \beta_n w(t)$$

?

$$\begin{cases} \beta_0 = b_0 \\ \beta_1 = b_1 - a_1\beta_0 \\ \vdots \\ \beta_n = b_n - a_1\beta_{n-1} - \ldots - a_{n-1}\beta_1 - a_n\beta_0 \end{cases}$$

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ -a_0 & -a_1 & \ldots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} w \\ y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta_0 w \end{cases}$$

Example:
Finding the state equations for $G(s)=\dfrac{12s+59}{s^2+6s+8}$

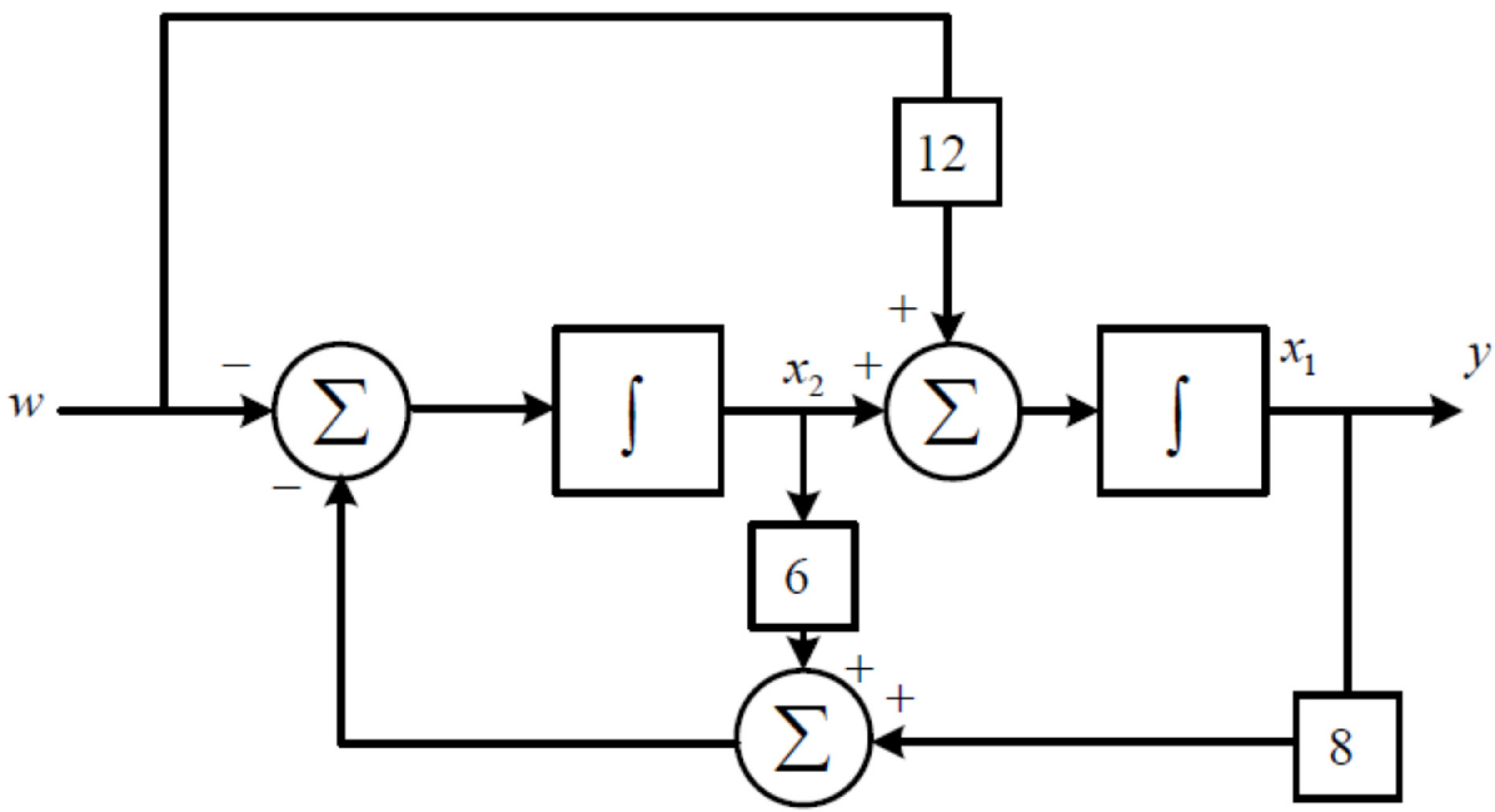
$$G(s)=\frac{\overbrace{12}^{b_1}s+\overbrace{59}^{b_2}}{s^2+\underbrace{6}_{a_1}s+\underbrace{8}_{a_2}}=\frac{Y(s)}{W(s)}$$

$$\rightarrow (s^2+6s+8)Y(s)=s^2Y(s)+6sY(s)+59Y(s)=(12s+59)W(s)=12s\,W(s)+59W(s)$$

$$\xrightarrow{L^{-1}\{\}}\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+8y(t)=12\frac{dw(t)}{dt}+59w(t)$$

$$n=2\rightarrow\begin{cases}\beta_0=b_0=0\\ \beta_1=b_1-a_1\beta_0=12\\ \beta_2=59-6\times12=-13\end{cases}$$

$$\begin{cases}x_1(t)=y(t)\\ x_2(t)=\dot{x}_1(t)-12w(t)\\ \dot{x}_2(t)=-8x_1(t)-6x_2(t)-w(t)\end{cases}\rightarrow\begin{cases}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -8 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}12\\ -13\end{bmatrix}w\\ y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+0\times w\end{cases}$$



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8